Two-sided problems with choice functions, matroids and lattices

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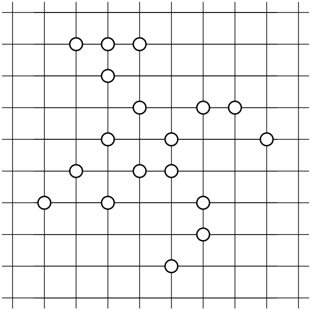
A competition problem

Prove that any finite subset H of the planar grid has a subset K with the property that

1. any vertical or horizontal line intersects K in at most 2 points,
2. any point of $H \setminus K$ lies on a vertical or horizontal segment determined by K.



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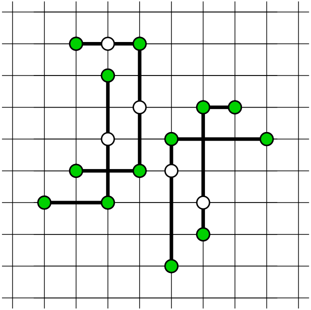

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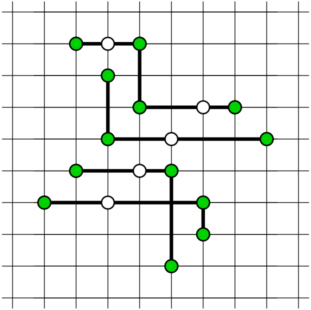

Two-sided markets: college admissions and graphs

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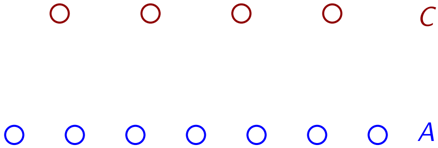

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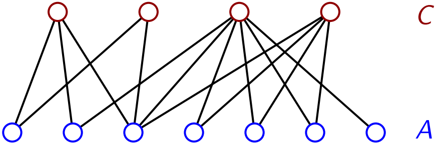

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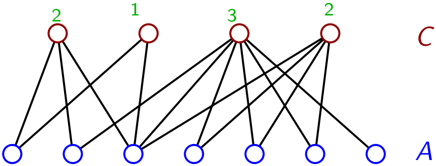

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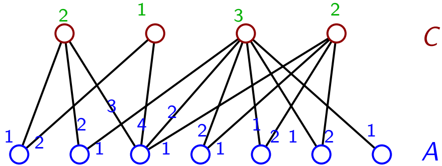

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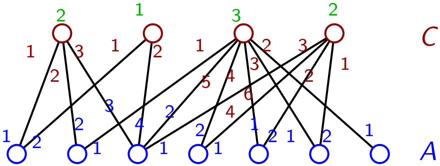

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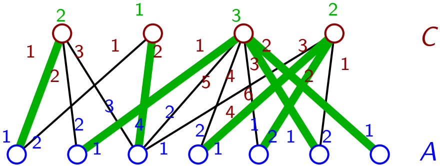

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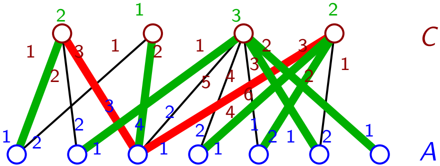

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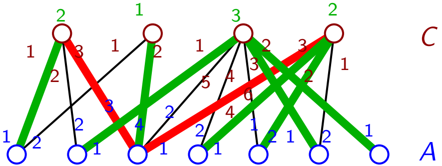

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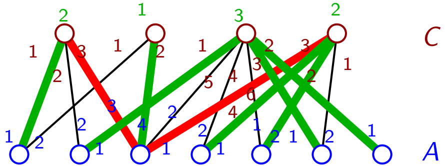

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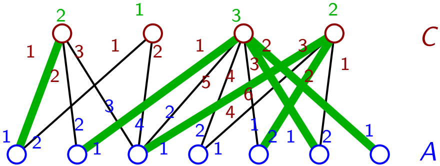

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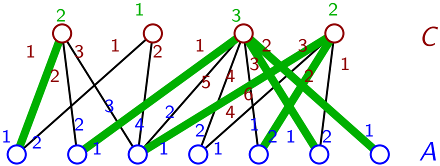

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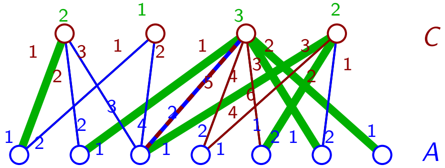

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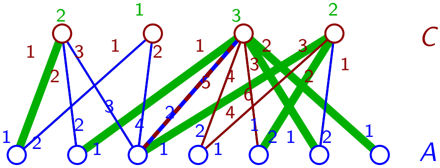

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If students are offered $S \cup \mathcal{D}_A(S)$ then they choose S ,
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That is, $\mathcal{C}_A(S \cup \mathcal{D}_A(S)) = S$ and $\mathcal{C}_C(S \cup \mathcal{D}_C(S)) = S$.

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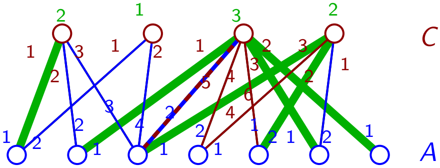

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Goal: A choice-function based approach to two-sided markets.

Stability and choice functions

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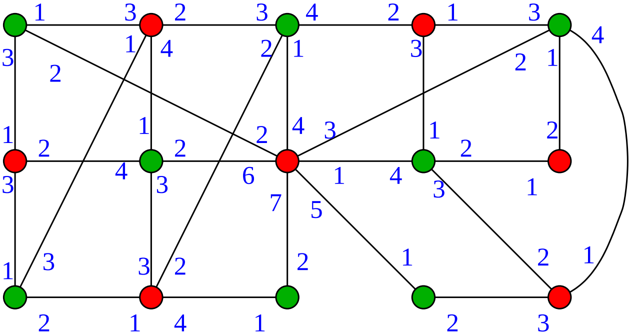
Fact: If $\mathcal{C}$ is substitutable and increasing then $\mathcal{C}$ is PI.

The deferred acceptance algorithm

Gale-Shapley Theorem: There always exists a stable matching.

The deferred acceptance algorithm



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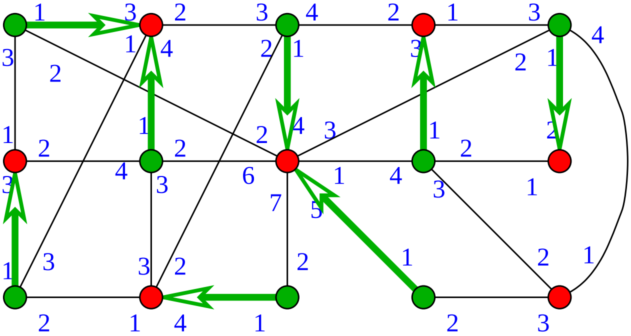

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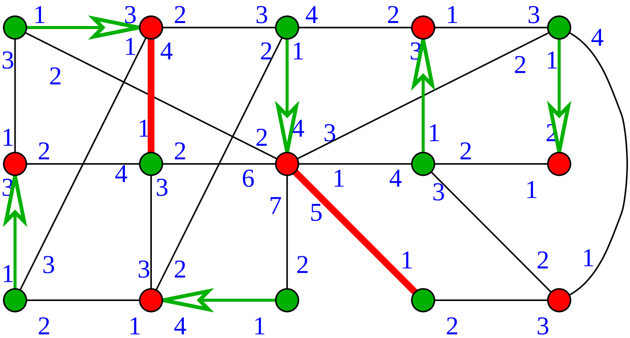

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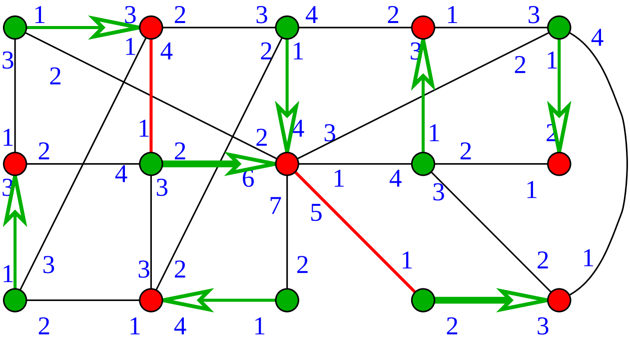

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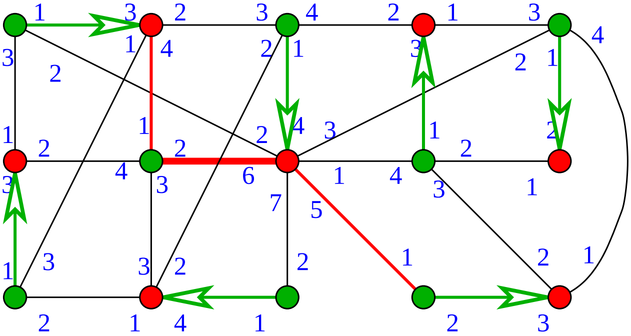

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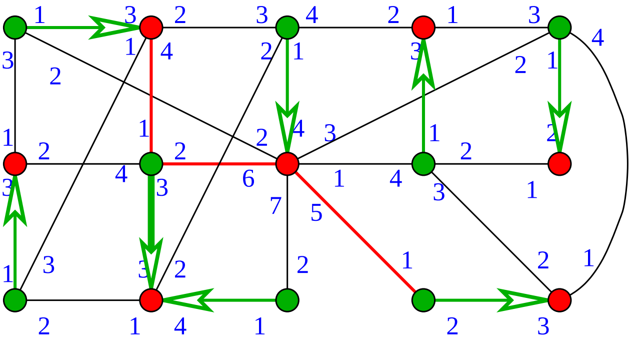

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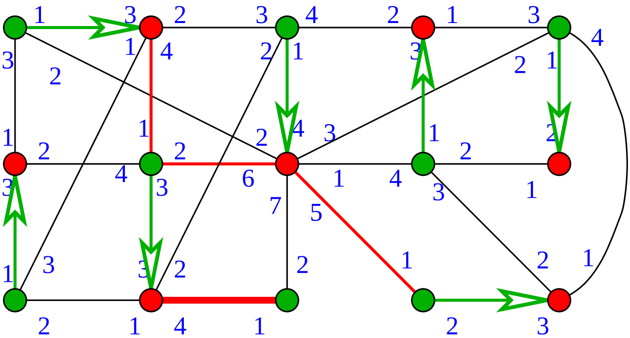

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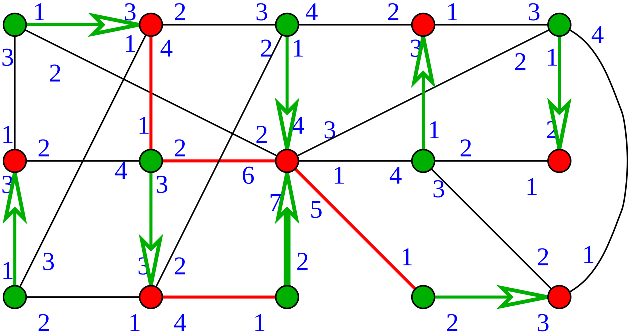

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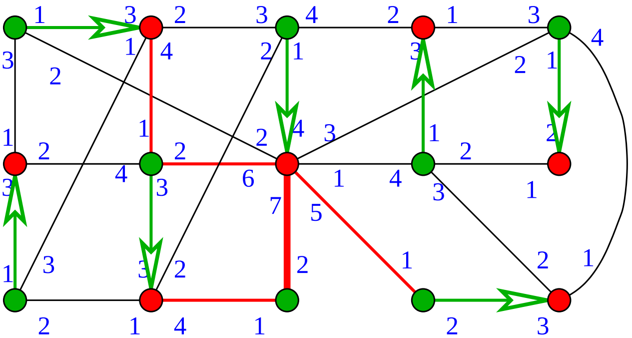

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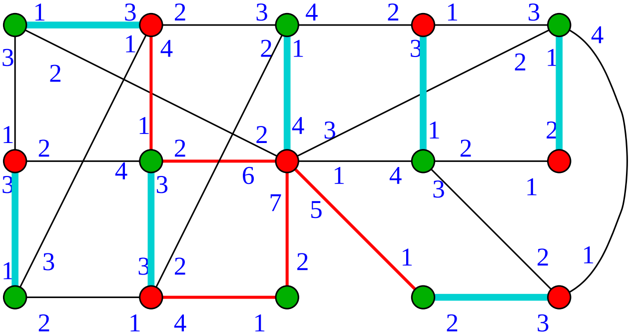

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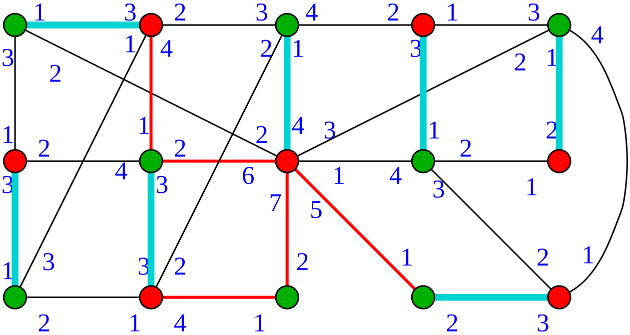

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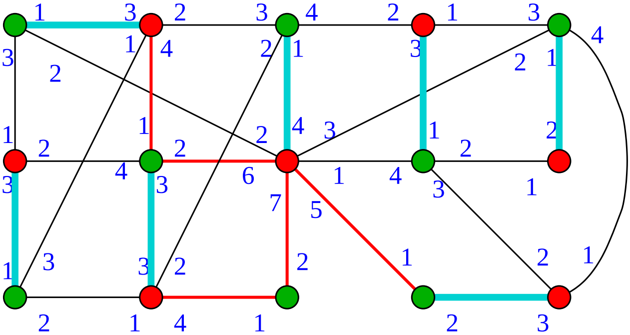

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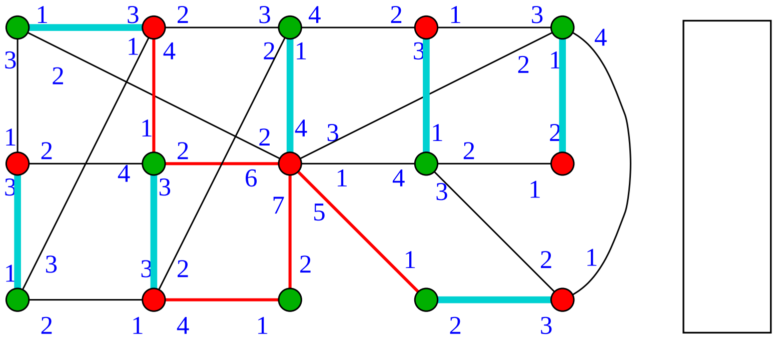

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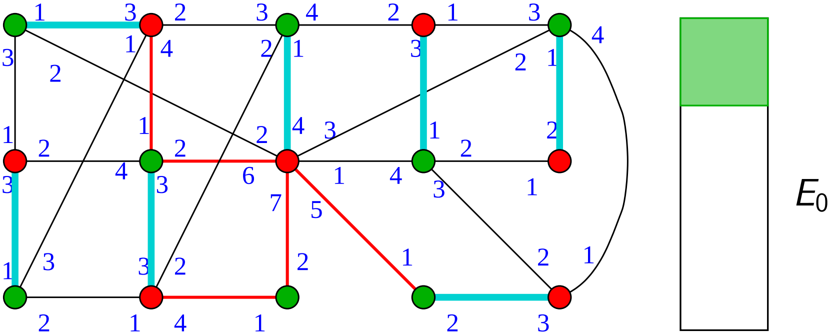

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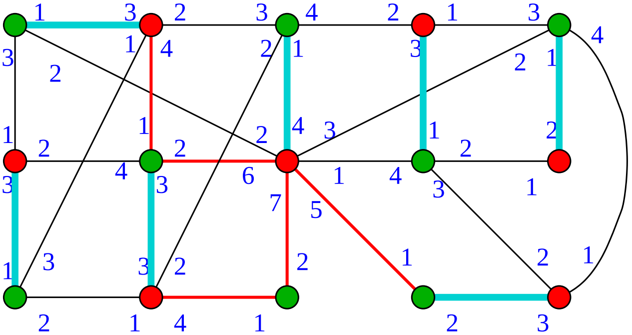

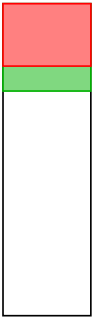

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Proof Boys propose, girls reject alternatingly until no rejection.

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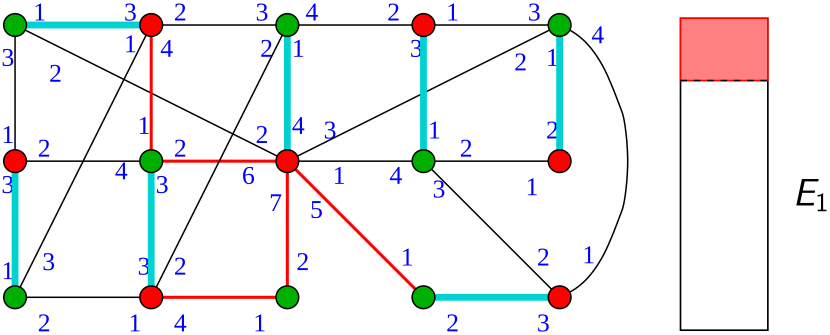

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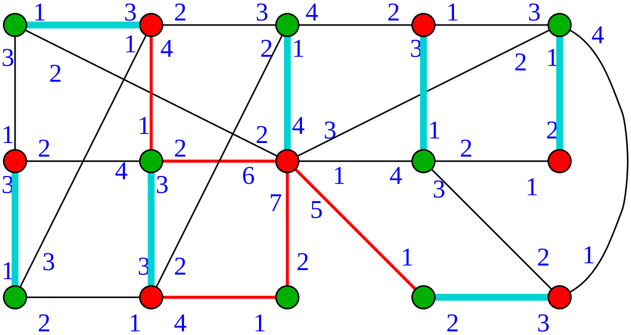

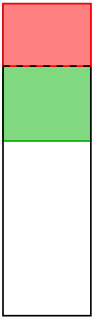

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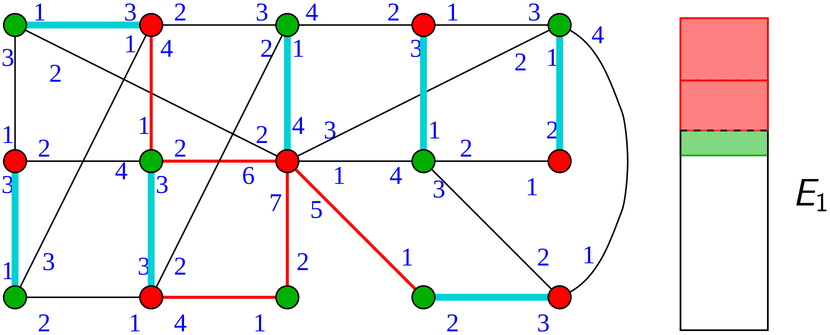

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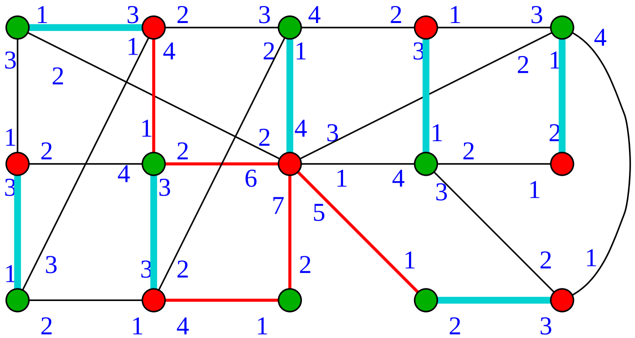

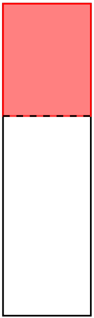

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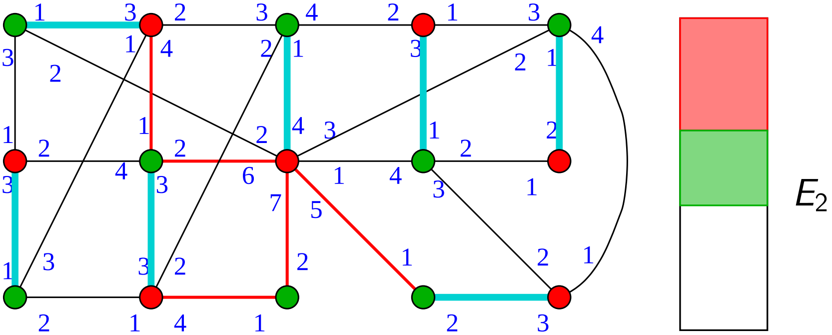

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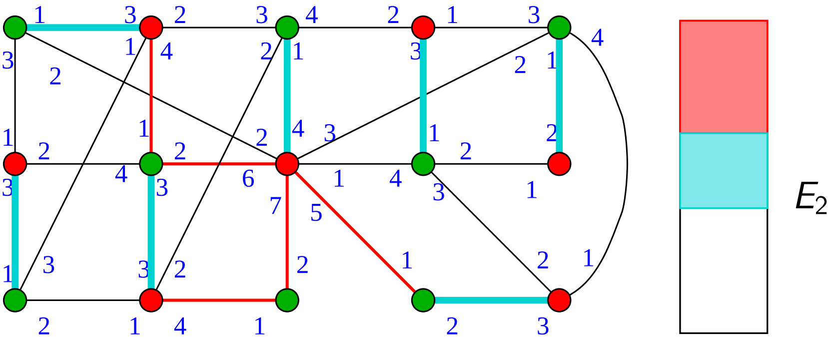

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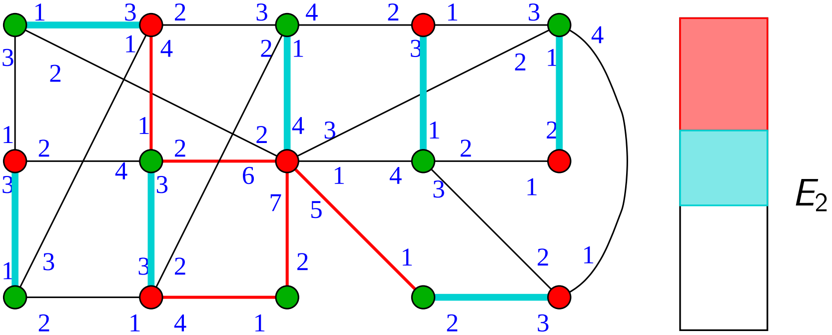

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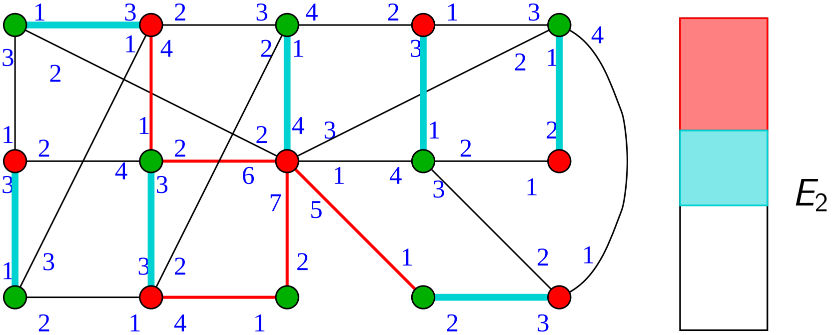

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Stupid question: What makes this algorithm work?

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Observation: The Gale-Shapely algorithm is an iteration of a monotone function. By definition, $E_{i+1} = \mathcal{F}(E_i)$, where
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Stronger lattice property: If both $\mathcal{C}_A$ and $\mathcal{C}_C$ are increasing and substitutable then lattice operations in Blair's thm are $S_1 \wedge S_2 = \mathcal{C}_A(S_1 \cup S_2)$ and $S_1 \vee S_2 = \mathcal{C}_C(S_1 \cup S_2)$.

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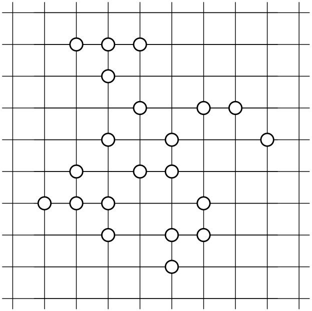
A special case



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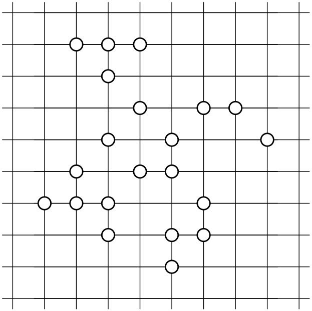

Rows=men, columns=women,

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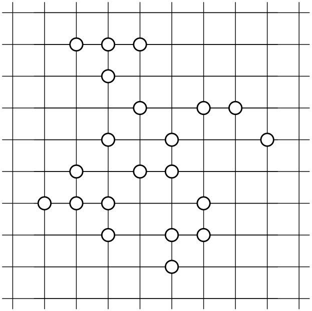

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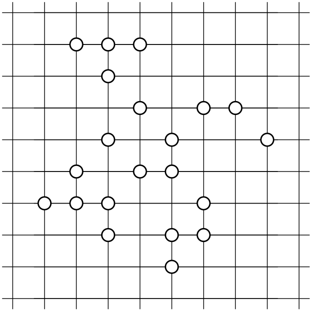

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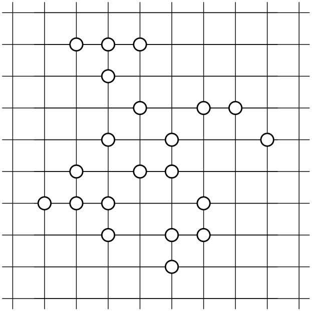

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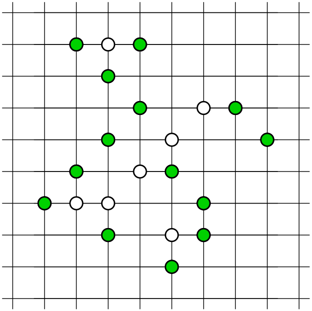

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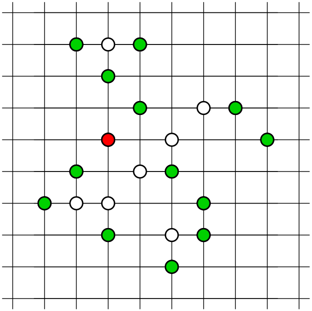

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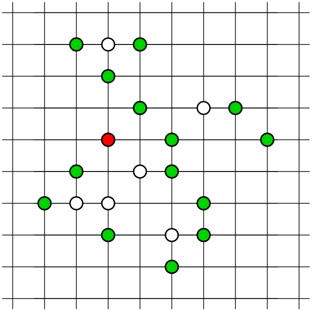

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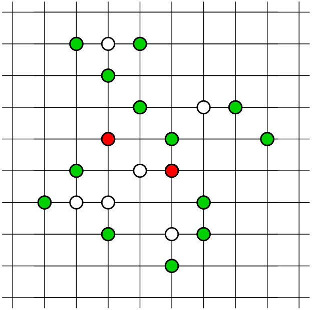

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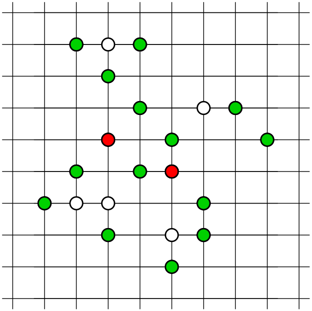

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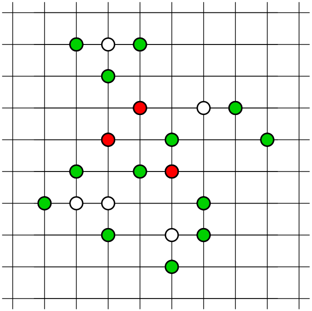

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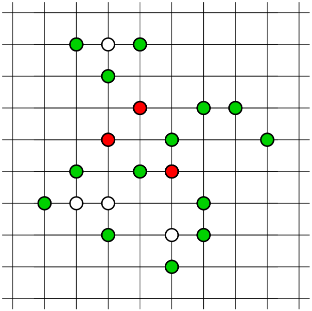

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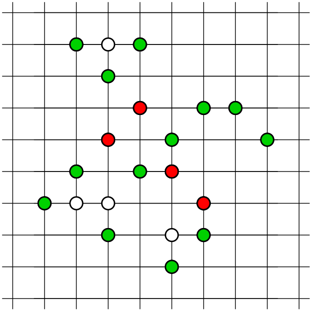

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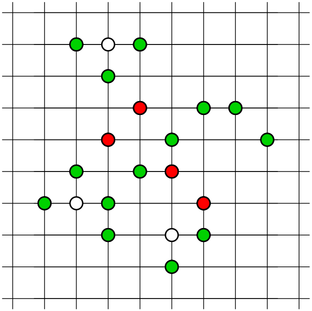

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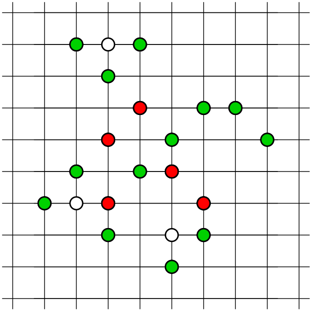

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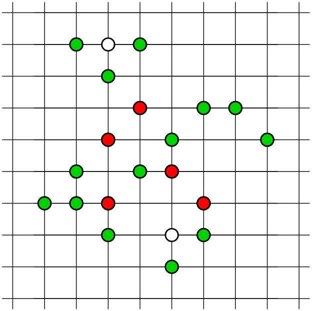

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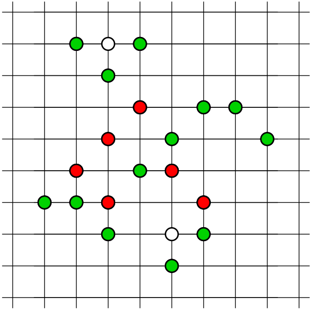

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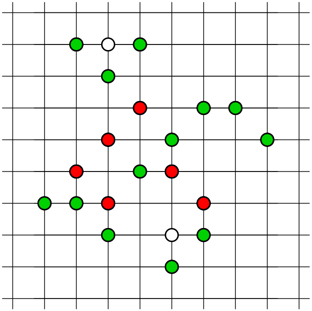

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The man-oriented GS algorithm finds the man-optimal stable solution: the "widest" set of gridpoints. The woman-optimal solution would be the "tallest" such set.

Stable assignments on many-to-one markets

Gale-Shapley: in the college admissions model (strict preferences and college-quotas) there always exists a stable assignment. (DA, college and student-optimality and lattice property.)
Hamada-Miyazaki-Iwama: if colleges have lower quotas as well then the number of blocking edges is inapproximable.
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NP-completeness: an efficient algorithm for the problem would imply an efficient algorithm for many truly difficult problems.

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Further, if no lower quotas, but common quotas for sets of colleges, then again, the problem is NP-complete.

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Main tool: matroid-based choice functions.

A crash course on matroids

Matroid: $\mathcal{M} = (E, \mathcal{I})$ st (1) $\emptyset \in \mathcal{I}$, (2) $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$, (3) $A, B \in \mathcal{I}$, $|A| < |B| \Rightarrow \exists b \in B \setminus A : A \cup \{b\} \in \mathcal{I}$.

Examples: (1) Linear matroid (vectors with linear independence)

(2) Graphic matroid (edges of a graph with no cycles)

(3) Trivial matroid ($\mathcal{I} = 2^E$)

(4) Uniform matroid truncation of a trivial matroid

(5) Partition matroid

($E = E_1 \cup E_2 \cup \ldots \cup E_k$ is a partition. $I \in \mathcal{I}$ iff $|I \cap E_i| \leq 1$).

(6) Direct sum of uniform matroids ($E = E_1 \cup E_2 \cup \ldots \cup E_k$ is a partition, $b_1, b_2, \ldots, b_k$ given. $I \in \mathcal{I}$ iff $|I \cap E_i| \leq b_i \forall i$).

Basis: maximal independent set of E (same cardinality)

Rank fn: $rk(A) = \max\{|A'| : A' \subseteq A \text{ independent}\}$.

Span: $sp(A) := \{e \in E : rk(A \cup \{e\}) = rk(A)$.

Greedy prop: maxweight indep set can be constructed greedily deciding on the elements one by one in the order of decr weights.

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"Rural hospitals" Thm: If both $\mathcal{C}_C$ and $\mathcal{C}_A$ are greedy choice fn's then stable assignments have the same span.

The classified stable matching problem

Problem input: Two-sided market between C and A with set E of possible contracts, nested systems $\mathcal{Q}_C, \mathcal{Q}_A \subseteq 2^E$ of common quota sets, $l, u : \mathcal{Q}_A \cup \mathcal{Q}_A \to \mathbb{N}_+$ lower and upper quotas and preferences $\prec_C$ and $\prec_A$ st any common quota set is linearly ordered.

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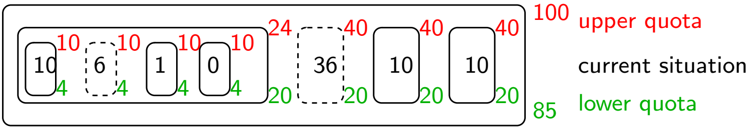
Solution: Application of the choice function framework.

Key question: how do colleges decide on accepted contracts if contracts are coming in the order of preference.

Colleges' choice function



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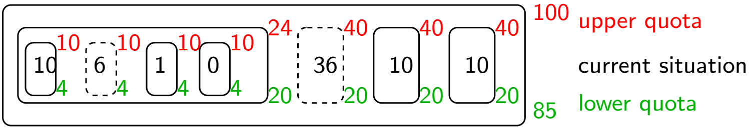

Obs: Dashed quota sets are “implicitely” saturated, no new contract is possible.

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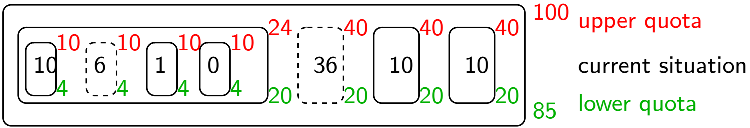

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Recursive definition: For $F \subseteq E$, if Q is an inclwise min member of $\mathcal{Q}_C$ then $d(Q, F) := \max\{|F \cap Q|, l(Q)\}$.

If $Q \in \mathcal{Q}_C$ has maximal children $Q_1, \ldots Q_k$ then

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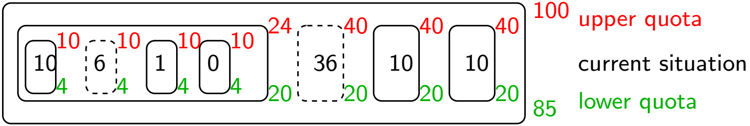

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Key thm: Family $\mathcal{I}_C := \{F \subseteq E : d(Q,F) \leq u(Q) \quad \forall Q \in \mathcal{Q}_C\}$ forms the independent sets of a matroid.

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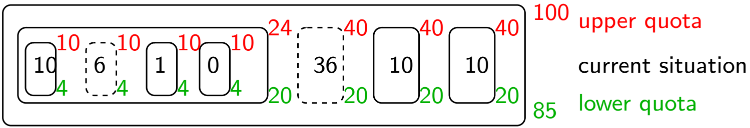

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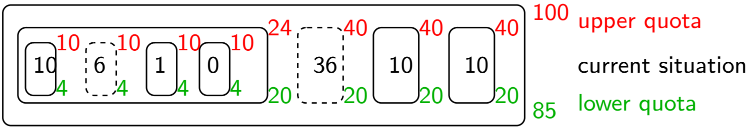
Key thm: Family $\mathcal{I}_C := \{F \subseteq E : d(Q,F) \leq u(Q) \quad \forall Q \in \mathcal{Q}_C\}$ forms the independent sets of a matroid.

Cor: Stable assignment for ch fns $\mathcal{C}_C$ and $\mathcal{C}_A$ always exists.

Colleges' choice function



Obs: Dashed quota sets are "implicitely" saturated, no new contract is possible.

Recursive definition: For $F \subseteq E$, if Q is an inclwise min member of $\mathcal{Q}_C$ then $d(Q,F) := \max\{|F \cap Q|, l(Q)\}$.

If $Q \in \mathcal{Q}_C$ has maximal children $Q_1, \ldots Q_k$ then

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Trick: As span is always the same, either all $\mathcal{C}_C\mathcal{C}_A$-stable solutions obey the lower quotas or none of them does. So if Gale-Shapley solution violates a lower quota then no stable assignment exists whatsoever. Otherwise GS outputs a solution.

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- **Lesson for Economists**:
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- **Lesson for Mathematicians**:
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Thank you for the attention!