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Brief Introduction to Discrete Convex Analysis

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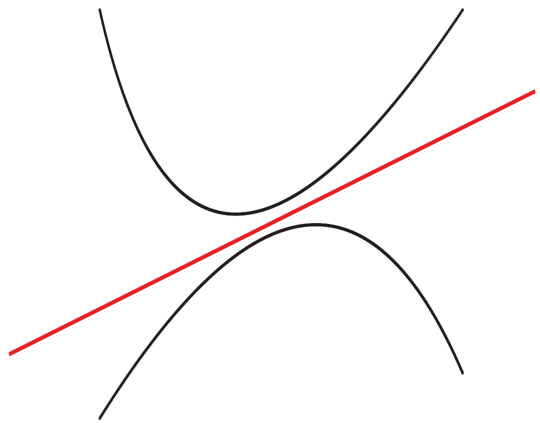
$$f: \mathbf{Z}^n \to \mathbf{Z}$$
$$f: \mathbf{Z}^n \to \mathbf{R}$$
$$f: \mathbf{R}^n \to \mathbf{R}$$



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Connections

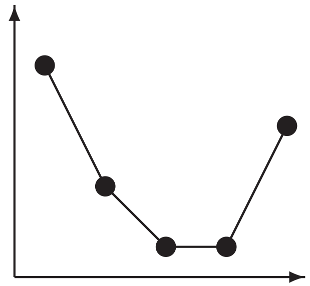
Game/Economics		Discrete Convex Analysis
convex game	$\longleftrightarrow$	supermodular set func
decreasing marginal return	$\longleftrightarrow$	concave/submodular
gross substitutes	$\longleftrightarrow$	$\mathrm{M}^{\natural}$-concave

Discrete Convex Analysis

Convexity Paradigm in Discrete Optimization

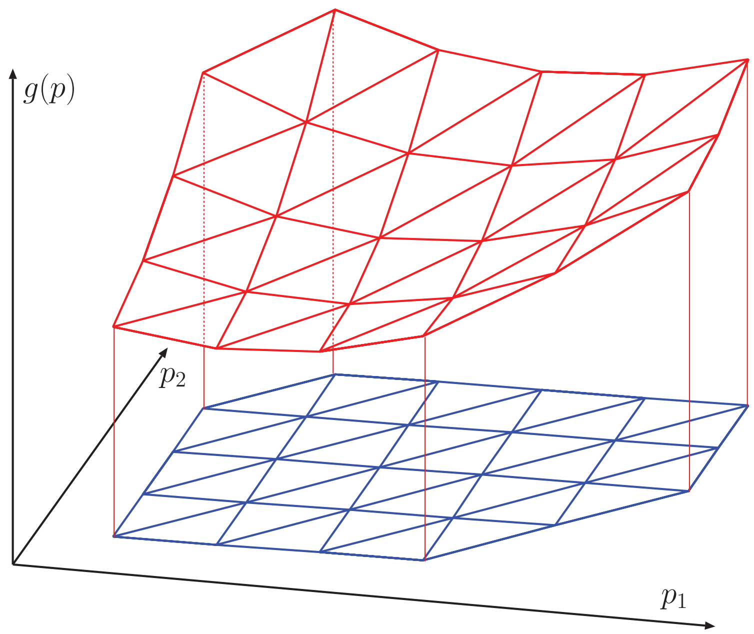
Matroid Theory + Convex Analysis

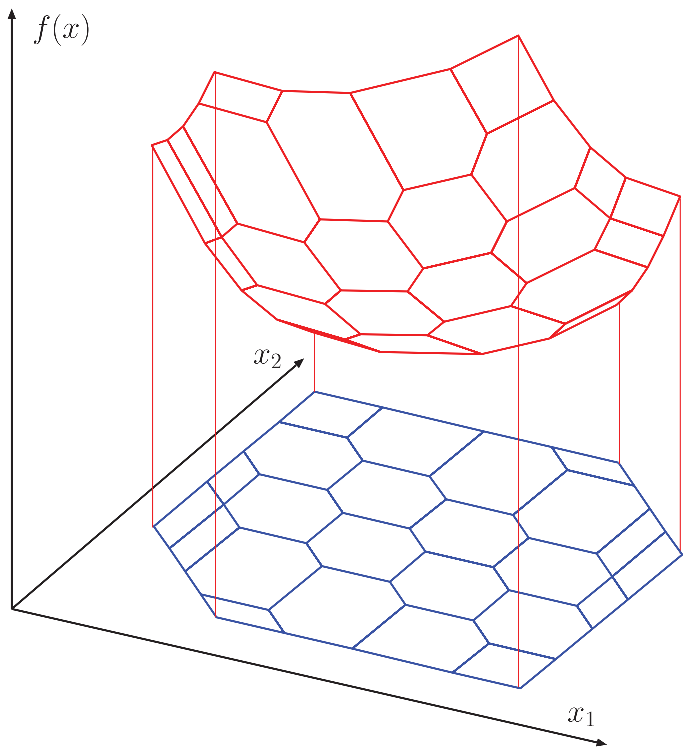
Submodular fn	$\leftrightarrow$	Matroid base
$\Downarrow$		$\Downarrow$
L-convex fn	$\leftrightarrow$	M-convex fn



- Global optimality $\Longleftrightarrow$ local optimality
- Conjugacy: Legendre–Fenchel transform
- Duality (Fenchel min-max, discrete separation)
- Minimization algorithms
- Applications: OR, game, economics, matrices

Discrete Convex Functions





L$^\natural$-convex fn **M$^\natural$-convex fn**

Some History

1935	Matroid	Whitney, Nakasawa
1965	Submodular function	Edmonds
1975	Application of matroid	Iri, Recski
1983	Submodularity and convexity	Lovász, Frank, Fujishige
1990	Valuated matroid	Dress–Wenzel
	Integrally convex fn	Favati–Tardella
1996	Discrete convex analysis	Murota
2000	Submod. fn minimization algorithm	Iwata–Fleischer–Fujishige, Schrijver

Contents

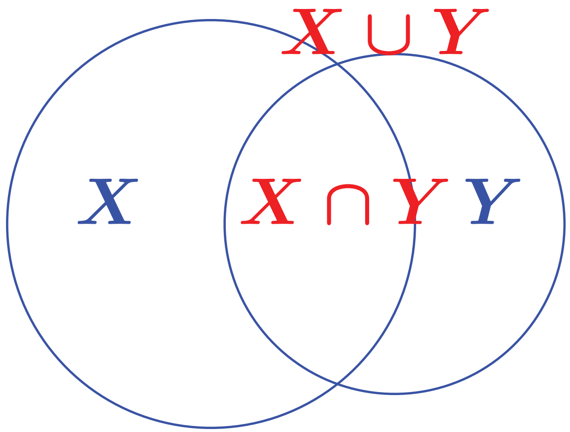
B1.

Submodularity and Convexity

Submodular Function

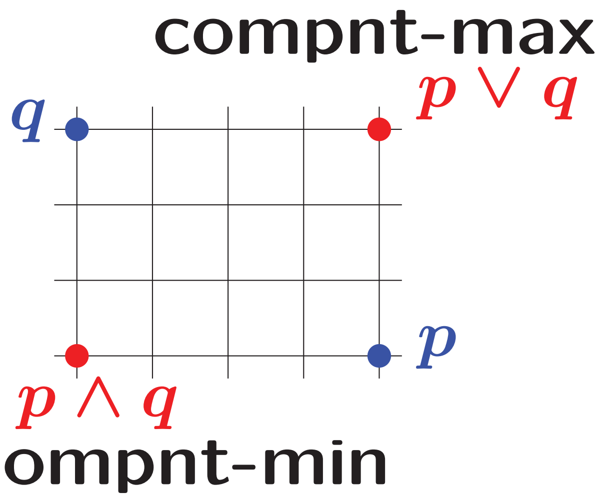
Set function ρ is submodular:

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$



$g : \mathrm{Z}^n \to \mathrm{R}$ is submodular:

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$



Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms

(Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

min $\Rightarrow$ polynomial, max $\Rightarrow$ NP-hard

- Convex extension (Lovász)

set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

discrete separation, Fenchel min-max

Duality for submodular set functions = Convexity + Discreteness

On the other hand ...

decreasing
marginal return $\longleftrightarrow$ concave/submodular

This means ...

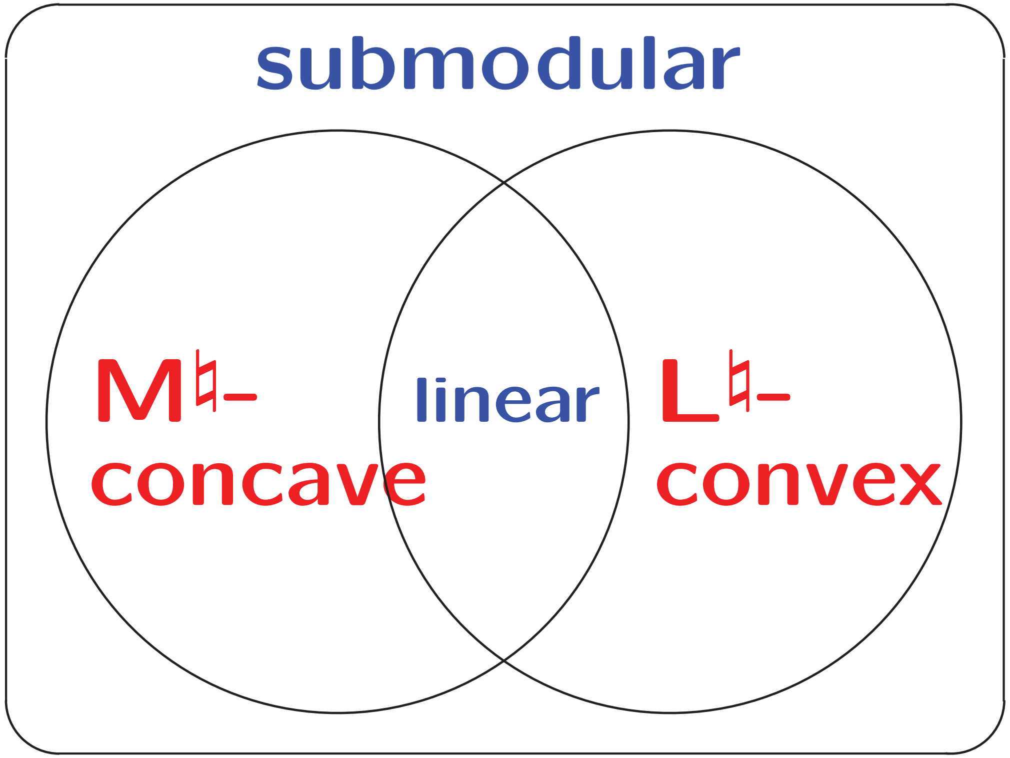
Submodular $\approx$ Concave

Moreover ...

$\rho(X) = \varphi(|X|)$ (φ: concave) is submodular

Submodularity & Convexity in DCA

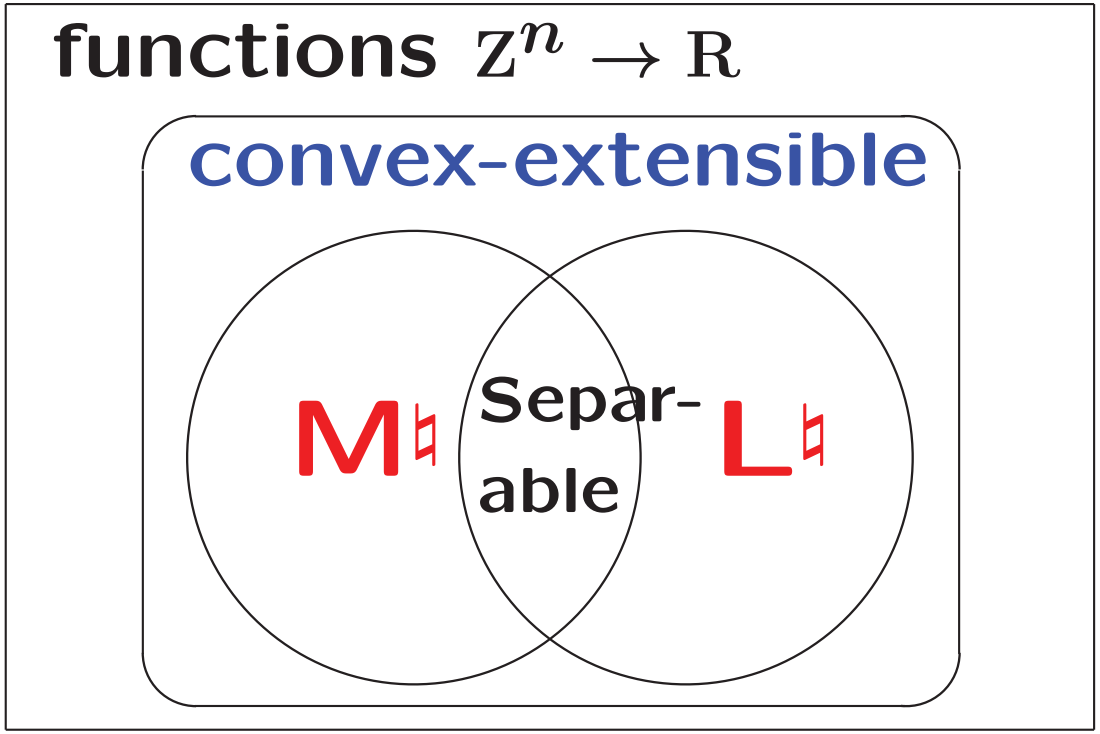
- M$^\natural$-concave function is submodular
- L$^\natural$-convex function is submodular



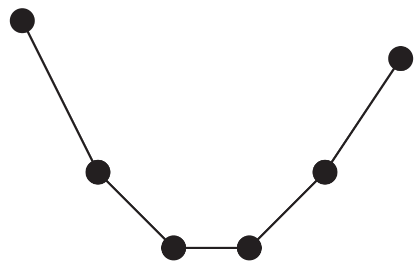
B2.

L-convex and M-convex Functions

Discrete Convex Functions



$f : \mathbb{Z}^n \to \mathbb{R}$



f is convex-extensible

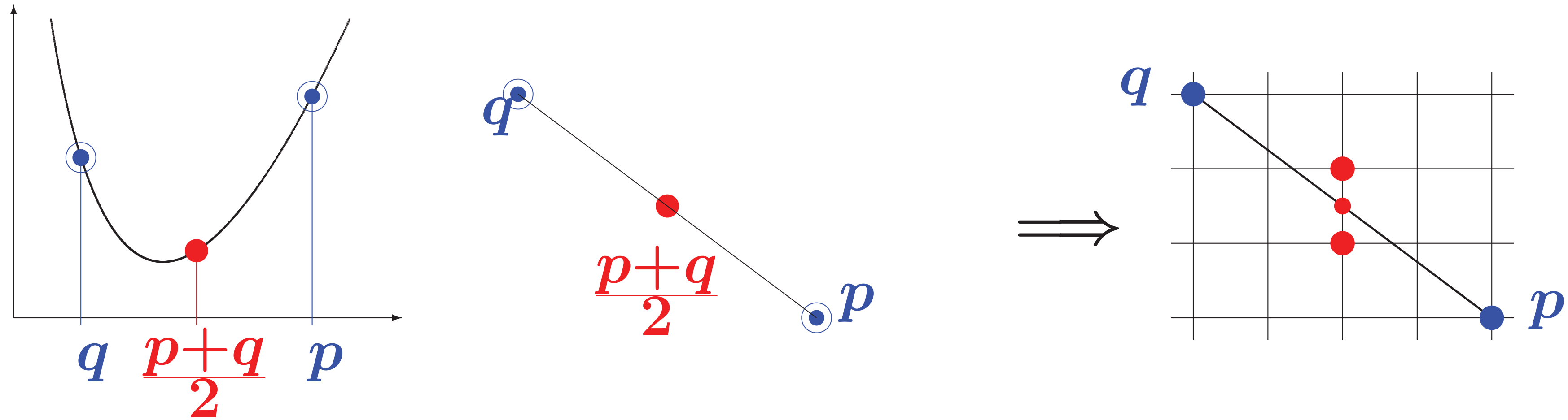
$\Leftrightarrow$ $\exists$ convex $\overline{f}$:

$f(x) = \overline{f}(x)$

Convex-extensibility does not help much

L♮-convexity from Mid-pt-convexity

(Murota 98, Fujishige–Murota 00)



Mid-point convex ($g : \mathbf{R}^n \to \mathbf{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

$\Longrightarrow$ **Discrete mid-point convex** ($g : \mathbf{Z}^n \to \mathbf{R}$)

$$g(p) + g(q) \geq g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right)$$

L♮-convex function (L = Lattice)

L♮-convexity from Submodularity

—Original definition of L♮-convexity—

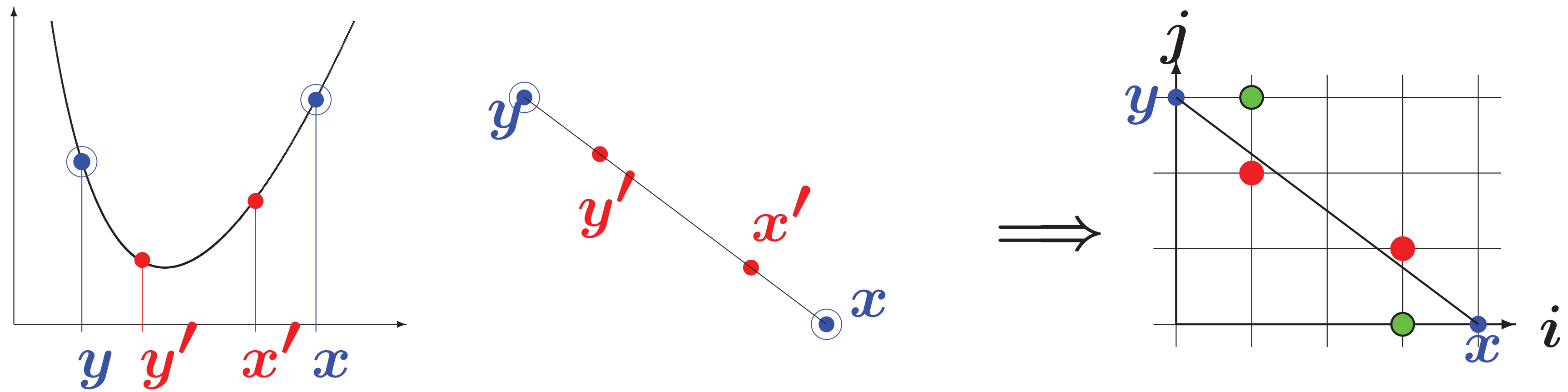
Def: $g : \mathbf{Z}^n \to \mathbf{R}$ is L♮-convex $\iff$

$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ is submodular in (p_0, p)

$\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R}$, $\quad \mathbf{1} = (1, 1, \ldots, 1, 1)$

M♮-convexity from Equi-dist-convexity

(Murota 96, Murota–Shioura 99)



Equi-distance convex ($f : \mathbb{R}^n \to \mathbb{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x - y)) + f(y + \alpha(x - y))$$

$\Longrightarrow$ **Exchange** ($f : \mathbb{Z}^n \to \mathbb{R}$) $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \left[f(x - e_i) + f(y + e_i), \right.$$

$$\left. \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \right]$$

M♮-convex function (M = Matroid)

Gross Substitutes (for set function)

$f : 2^V \to \mathrm{R}$ utility (reservation value) function

p price vector

$D(p) = \arg\max(f - p) = \{X \mid f(X) - p(X)$ is maximum$\}$

demand correspondence

Gross substitutes property: (Kelso–Crawford 82)

$X \in D(p), \quad p \leq q$

$\Rightarrow \quad \exists\, Y \in D(q) \ : \ \{i \in X \mid p_i = q_i\} \ \subseteq \ Y$

Equiv. cond. for $D(p)$ (Gul–Stacchetti 99)

Equiv. cond. for f (Reijnierse–van Gallekom–Potters 02)

& equivalence to M$^\natural$-concavity (Fujishige–Yang 03)

$\Longrightarrow$ To be extended for $f : \mathrm{Z}^n \to \mathrm{R}$

Gross Substitutes for f (not for $D(p)$)

$f : 2^V \to \mathrm{R}$ (set function)

f: **gross substitutes** $\iff$

(i) $f(S \cup \{i,j\}) + f(S) \leq f(S \cup \{i\}) + f(S \cup \{j\})$

(submodular)

(ii) $f(S \cup \{i,j\}) + f(S \cup \{k\}) \leq$

$\max[f(S \cup \{i,k\}) + f(S \cup \{j\}), f(S \cup \{j,k\}) + f(S \cup \{i\})]$

(Reijnierse–van Gallekom–Potters 02)

cf. Local exchange axiom of M$^\natural$-concave functions

Gross Substitutes = M$^\natural$-concavity

$f : \mathbf{Z}^n \to \mathbf{R}$ (function on integer lattice)

Gross substitutes for $f : \mathbf{Z}^n \to \mathbf{R}$:

$x \in \arg\max(f - p + p_0 \mathbf{1})$,

$p \leq q$, $p_0 \leq q_0$, $\arg\max(f - q + q_0 \mathbf{1}) \neq \emptyset$

$\Rightarrow$ $\exists\, y \in \arg\max(f - q + q_0 \mathbf{1})$: $y_i \geq x_i$ if $p_i = q_i$

$y_1 + \cdots + y_n \leq x_1 + \cdots + x_n$ if $p_0 = q_0$

Gross substitutes $\iff$ M$^\natural$-concave

(Fujishige-Yang, Danilov-Koshevoy-Lang, Murota-Tamura 03, Milgrom-Strulovici 09)

$\Longrightarrow$ Applications to economics/game theory

M$^\natural$-convex Function: Examples

Quadratic: $f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow \ a_{ij} \geq 0, \qquad a_{ij} \geq \min(a_{ik}, a_{jk}) \ \ (\forall k \not\in \{i,j\})$$

Min value: $f(X) = \min\{a_i \mid i \in X\}$ [unit preferene]

Matroid rank: $f(X) =$ rank of X

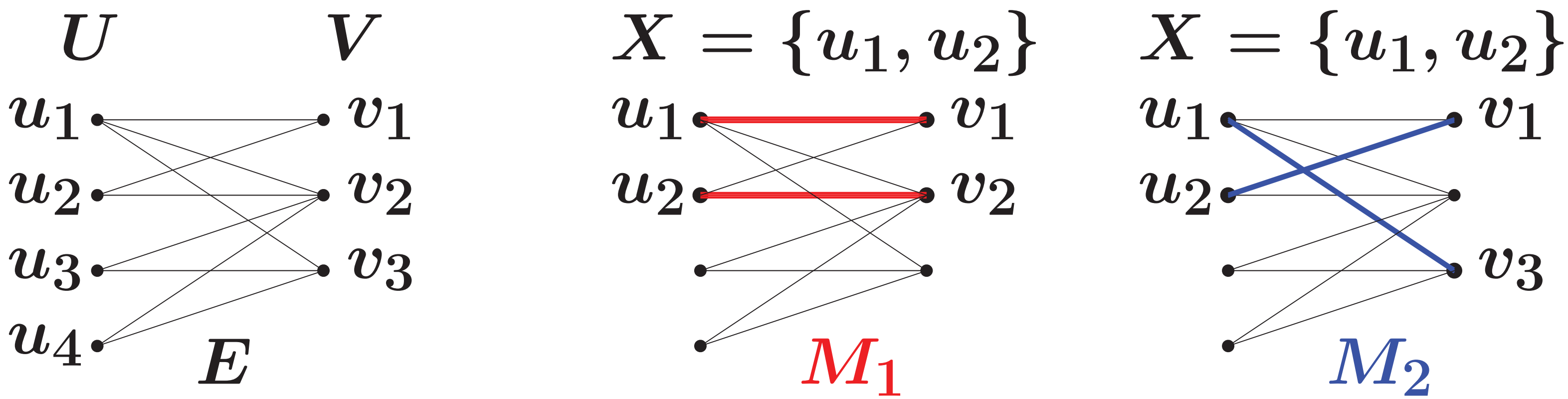
Cardinality convex: $f(X) = \varphi(|X|)$ (φ: convex)

Separable convex: $f(x) = \sum_{i=1}^{n} \varphi_i(x_i)$ (φ_i: convex)

Laminar convex: $f(x) = \sum_{A} \varphi_A(x(A))$ (φ_A: convex)

$\{A, B, ...\}$: laminar $\Leftrightarrow$ $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

Matching / Assignment



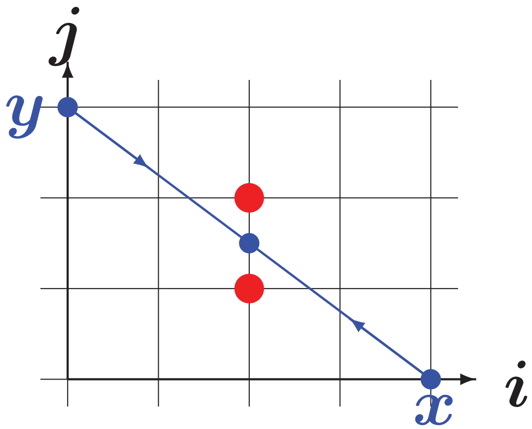
Max weight for $X \subseteq U$ **(w: given weight)**

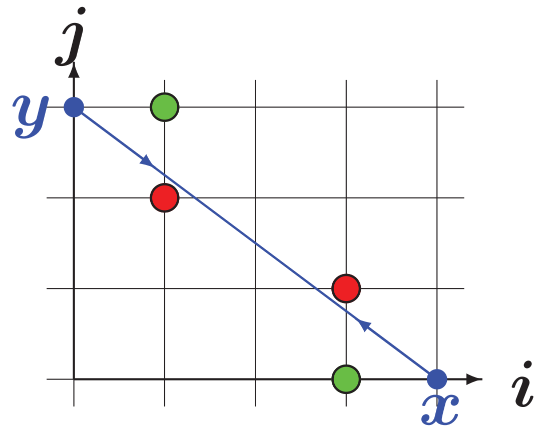
$$f(X) = \max\{\sum_{e \in M} w(e) \mid M\text{: matching}, \ U \cap \partial M = X\}$$

Max-weight func f is M$^\natural$-concave (Murota 96)

Summary: Continuous to Discrete

Continuous $\mathbf{R}^n \to \mathbf{R}$		Discrete $\mathbf{Z}^n \to \mathbf{R}$
mid-pt convex	$\longrightarrow$	disc mid-pt convex
$\Updownarrow$	discr	($L^\natural$-convex)
convex		
$\Updownarrow$	discr	($M^\natural$-convex)
equi-dist convex	$\longrightarrow$	exchange property





B3.

Conjugacy and Duality

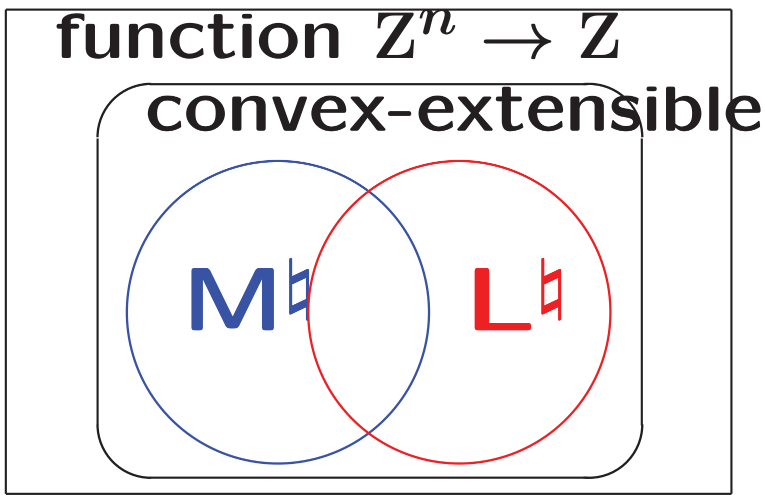
M-L Conjugacy Theorem

Integer-valued discrete fn $\quad f : \mathbf{Z}^n \to \overline{\mathbf{Z}}$

Legendre transform: $\quad f^\bullet(p) = \sup_{x \in \mathbf{Z}^n} [\langle p, x \rangle - f(x)]$

M♮-convex and L♮-convex are conjugate

$f \mapsto f^\bullet = g \mapsto g^\bullet = f$ (Murota 98)

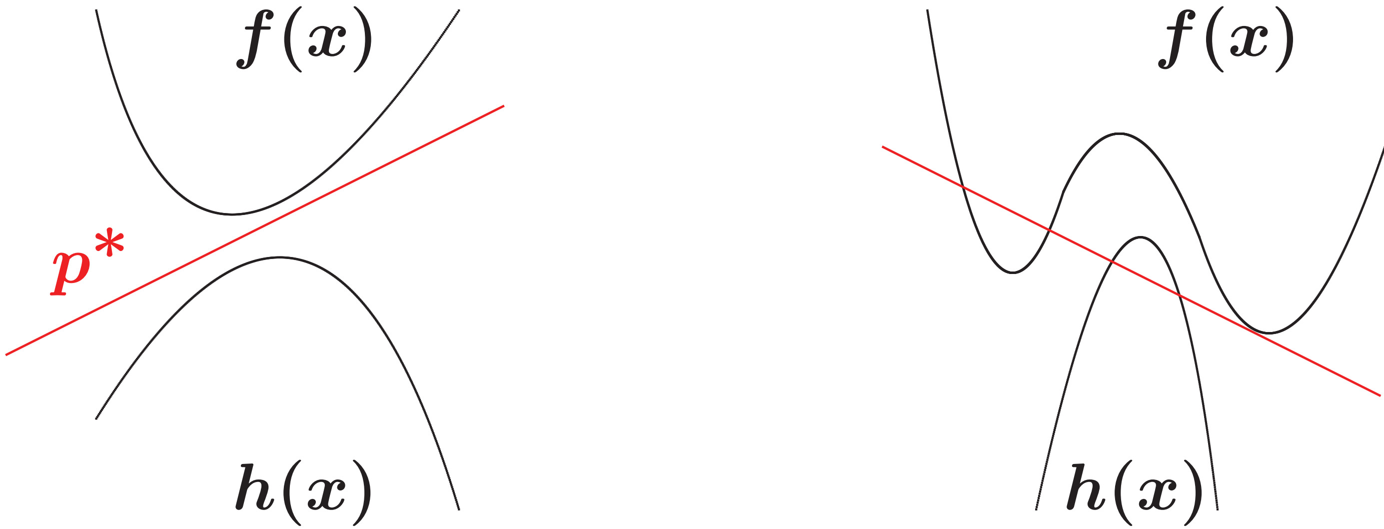


History of Discrete Conjugacy

Matroid bases	$\longleftrightarrow$	Matroid rank fn
Whitney 35		Whitney 35
$\Downarrow$		$\Downarrow$
Polymatroid	$\longleftrightarrow$	**Submodular fn**
Edmonds 70		Edmonds 70
$\Downarrow$		$\Downarrow$
Valuated matroid	|	Lovász extension
Dress–Wenzel 90	|	Lovász 83
$\Downarrow$	|	$\Downarrow$
	|	Submod. integ. conv. fn
	|	Favati-Tardella 90
		$\Downarrow$
M-convex fn	$\longleftrightarrow$	**L-convex fn**
Murota 96		Murota 98
$\Updownarrow$		$\Updownarrow$
$M^\natural$-convex fn	$\longleftrightarrow$	$L^\natural$-convex fn
Murota-Shioura 99		Fujishige–Murota 00

Duality: Separation Theorem

main issue in convexity paradigm



$f : \mathbb{R}^n \to \mathbb{R}$ **convex**
$h : \mathbb{R}^n \to \mathbb{R}$ **concave**

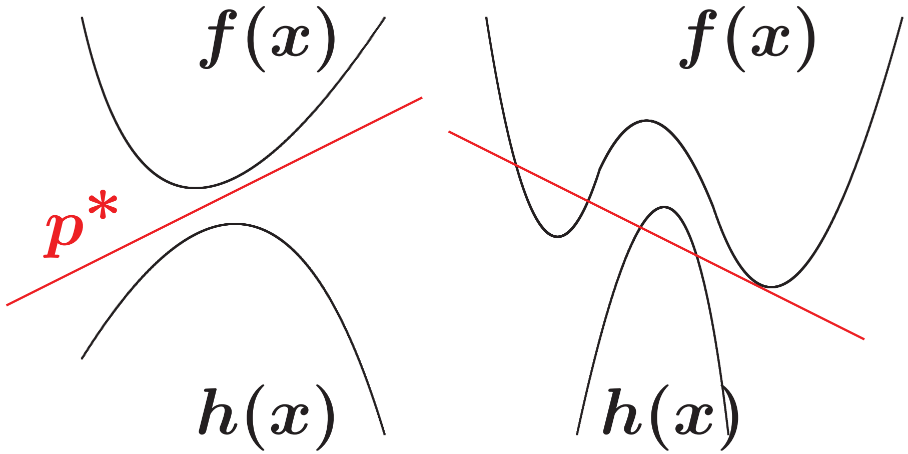
$$f(x) \geq h(x) \quad (\forall x) \Rightarrow \exists\ \alpha^* \in \mathbb{R},\ \ \exists\ p^* \in \mathbb{R}^n:$$

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbb{R}^n)$$

Discrete Separation Theorem

$f : \mathbf{Z}^n \to \mathbf{R}$ "convex"

$h : \mathbf{Z}^n \to \mathbf{R}$ "concave"



- $f(x) \geq h(x) \quad (\forall x \in \mathbf{Z}^n) \;\Rightarrow\; \exists\ \alpha^* \in \mathbf{R},\ \ \exists\ p^* \in \mathbf{R}^n$:

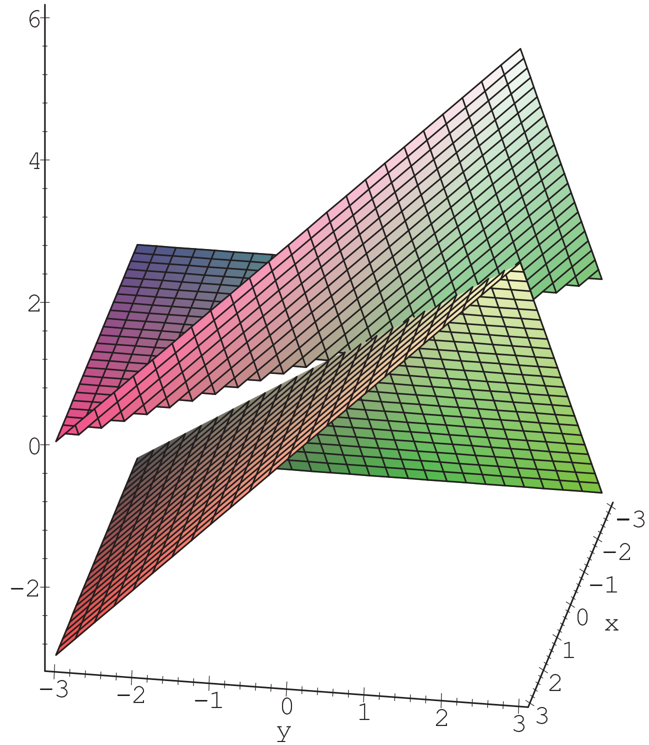
  $$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbf{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbf{Z},\ \ p^* \in \mathbf{Z}^n$

Difficulty of Discrete Separation (1)

$f(x, y) = \max(0, x + y)$ **convex**

$h(x, y) = \min(x, y)$ **concave**



separable but nonintegral

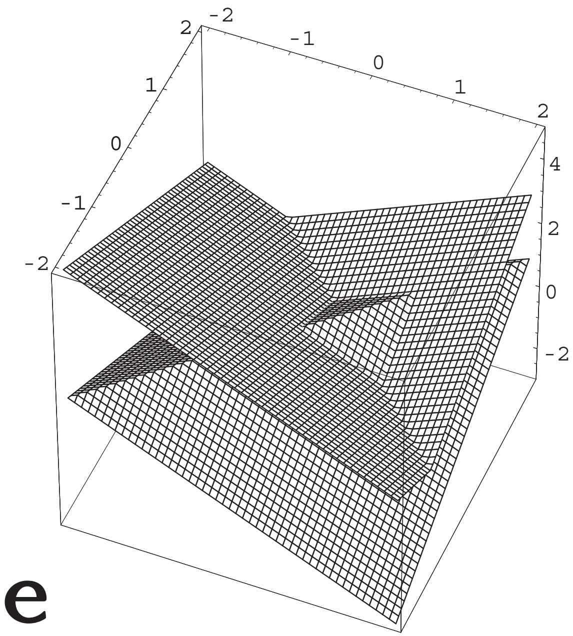
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

Difficulty of Discrete Separation (2)

Even real-separation is nontrivial

$f(x, y) = |x + y - 1|$ convex

$h(x, y) = 1 - |x - y|$ concave



- $f(x, y) \geq h(x, y)$ $(\forall (x, y) \in \mathrm{Z}^2)$ **true**
- $f(x, y) \geq h(x, y)$ $(\forall (x, y) \in \mathrm{R}^2)$ **not true**

since $f = 0 < h = 1$ at $(x, y) = (1/2, 1/2)$

$\Longrightarrow$ No $\alpha^* \in \mathrm{R}$, $p^* \in \mathrm{R}^2$ satisfies

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$$

Discrete Separation Theorems

(Murota 96/98)

M-separation Thm

$\supset$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 74, Frank 81)

(linear fn, indicator fn = $\mathrm{M}^\natural$-convex fn)

L-separation Thm

$\supset$ Discrete separation for submod. set function

(Frank 82)

(submod. set fn = $\mathrm{L}^\natural$-convex fn on 0–1 vectors)

Books

Murota: 離散凸解析，共立出版 2001

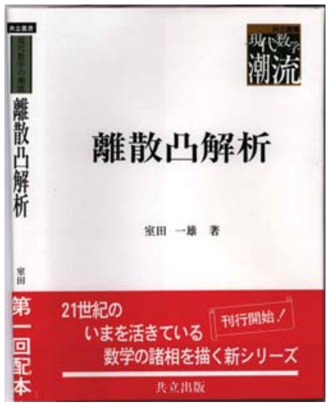
Murota: Discrete Convex Analysis, SIAM 2003

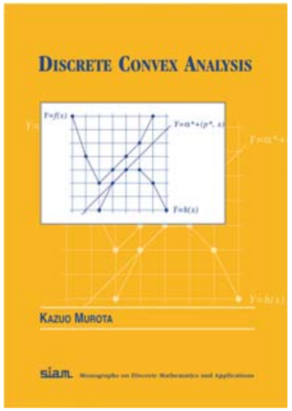
Fujishige: Submodular Functions and Optimization
2nd ed. (Chap. VII), Elsevier 2005

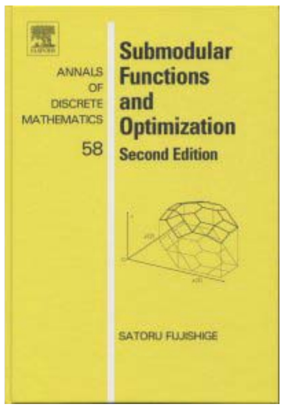
Murota: 離散凸解析の考えかた，共立出版 2007

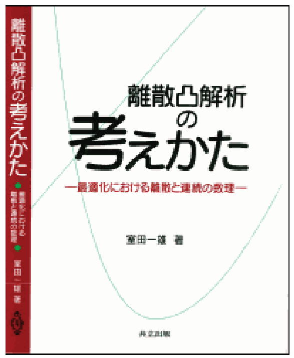
Tamura: 離散凸解析とゲーム理論，朝倉書店 2009
(Discrete Convex Analysis and Game Theory)

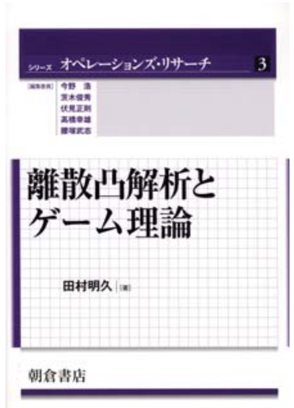
Murota–Shioura: 離散凸解析と最適化アルゴリズム，朝倉 2013

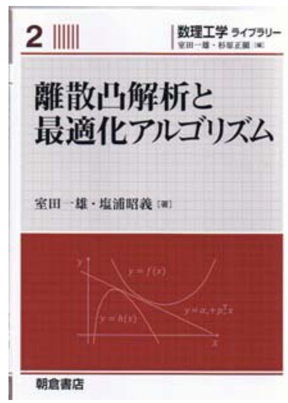












Survey/Slide/Video/Software

[Survey]
Murota: Recent developments in discrete convex analysis (Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, 219–260)

[Slide] [Video]

http://www.misojiro.t.u-tokyo.ac.jp/ murota/publist.html#DCA

[Video]

https://smartech.gatech.edu/xmlui/handle/1853/43257/

https://smartech.gatech.edu/xmlui/handle/1853/43258/

[Software] DCP (Discrete Convex Paradigm)
http://www.misojiro.t.u-tokyo.ac.jp/DCP/