Tradable Time-Specific Permits with Multiple Purchase Opportunity under Dynamic Uncertainty

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Outline

Motivation

- Tokyo Disneyland has hosted more than 13 million visitors for the last 5 years (14.8 million, 2012).3
- Some famous attraction (e.g. The Big Thunder Mountain, The Space Mountain, The Splash Mountain etc.) are overcrowded: guests must wait in line for long time. It is not uncommon to spend over 3 hours for one attraction in holiday seasons.
- To reduce the queuing loss, Disneyland provides the "Fastpass system" for overcrowded attractions: a "Fastpass" ticket for an attraction allows its holder to get ride the attraction in the pre-specified time windows (e.g. 10:40-11:40) immediately instead of waiting in a queue.

Motivation (cntd.)

- From the viewpoint of this workshop, the Fastpass system has a significant disadvantage:

Motivation (cntd.)

- From the viewpoint of this workshop, the Fastpass system has a significant disadvantage: there is no market mechanisms —— it is *subject to being unsold* and on a *first-come-first-served basis*.

Tradable Time-Specific Permits (TTP)

The Fastpass resembles a boarding pass and a concert ticket, each of which is a right that allows its holder to receive a certain service at the pre-specified time (referred to as the exercise time). If such right is traded in a markets, we call it as *tradable time-specific permits* (TTP).

We focus on the following two important features of the TTP:

1. TTPs could be purchased before its exercise time: e.g., a flight ticket on 3/1 can be purchased at 2/1 (28 days before), and 1/15 (45 days before) at different prices.
2. The value for the TTP of each buyer varies randomly with respect to time: e.g., the value for a flight ticket on 3/1 may fluctuate according to arrivals of the weather forecasts on the departure date.

Research Question

- How can we formulate the optimal assignment of the TTP with multiple purchase opportunity under dynamic uncertainty?
- Is it possible to achieve the social optimum assignment of the TTP with multiple purchase opportunity as an equilibrium of the markets with self-interested buyers?
- Is there any strategy-proof and individually rational mechanism that achieves the social optimal assignments?

Research Question

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- Is it possible to achieve the social optimum assignment of the TTP with multiple purchase opportunity as an equilibrium of the markets with self-interested buyers?
- Is there any strategy-proof and individually rational mechanism that achieves the social optimal assignments?→ may be next time.

Outline

Framework

Let us denote the fundamantal components:

$\mathcal{T} = \{0, 1, \cdots, \mathrm{T}\}$: the set of trading times, where $t = \mathrm{T}$ is the exercise time and $t = 0$ is the initial time.

$\mathcal{I}$: the set of buyers

Ω: the sample space (or, the set of sample paths)

Every random variable is defined as a map on Ω, e.g. a die roll is defined as a map $x : \Omega \to \{1, 2, 3, 4, 5, 6\} =: \mathcal{X}$. We may denote $\{x(\omega) : \omega \in \Omega\}$ as $\tilde{x}$.

$\mathcal{V}_i(\omega)$: buyer i's exercise value of the TTP on sample path $\omega \in \Omega$ (i.e. utility of receiving the service by exercise of the TTP). We assume that the exercise value could be negative (e.g. cancellation fee).

Example

- $\mathcal{T} = \{0, 1, 2\}$
- $\mathcal{I} = \{A, B, C\}$
- $\Omega = \{\omega_{FC}, \omega_{CF}, \omega_{CR}, \omega_{RC}\}$
- $\mathcal{V}(\omega) = (\mathcal{V}_A(\omega), \mathcal{V}_B(\omega), \mathcal{V}_C(\omega))$

At $t = 2$, each buyer uses a TTP (e.g. a flight ticket) for different activity: A visits City, B hikes Mountain and C goes home.

We assume that the value of the TTP equals to utility from each activity that may be affected by the weather at $t = 2$.

ω_{FC}: City: fine & Mountain: cloudy. $\mathcal{V}(\cdot) = (8, 6, 2)$

ω_{CF}: City: cloudy & Mountain: fine. $\mathcal{V}(\cdot) = (6, 8, 2)$

ω_{CR}: City: cloudy & Mountain: rain. $\mathcal{V}(\cdot) = (6, -4, 2)$

ω_{RC}: City: rain & Mountain: cloudy. $\mathcal{V}(\cdot) = (-4, 6, 2)$

C's activity is not affected by the weather.

Outline

Uncertainty and Information Structure

Uncertainty of buyers' exercise value is resolved with respect to time via increase in information.

$\mathcal{F}$: σ-algebra[2] on Ω (the most precise information).

$\mathcal{P}$: a probability measure on $(\Omega, \mathcal{F})$

We define the *information* that is available at the trade t as a σ-algebra on Ω, $\mathcal{F}(t) \subset \mathcal{F}$. It is natural to assume that the sequence of information is increasing[3]:

$$\mathcal{F}(0) \subseteq \mathcal{F}(1) \subseteq \cdots \subseteq \mathcal{F}(\mathrm{T}) \subseteq \mathcal{F}(0).$$

[2] A σ-algebra over Ω is a collection of subsets of Ω and Ω it self, which is closed under the complement and countable unions of its members.

[3] In other words, we define the information structure by a filtration.

Example (1)

- $\mathcal{T} = \{0, 1, 2\}$
- $\mathcal{I} = \{A, B, C\}$
- $\Omega = \{\omega_{FC}, \omega_{CF}, \omega_{CR}, \omega_{RC}\}$

At $t = 1$, a weather forecast arrivals

'No Rain' : the true sample path is in either $\Omega_{NR} = \{\omega_{FC}, \omega_{CF}\}$

'No Fine' : the true sample path is in $\Omega_{NF} = \{\omega_{CR}, \omega_{RC}\} = \Omega_{NR}^{\complement}$

$$\mathcal{F}(0) = \{\emptyset, \Omega\}, \quad \mathcal{F}(1) = \{\emptyset, \Omega_{NR}, \Omega_{NR}^{\complement}, \Omega\}$$

$$\mathcal{F}(2) = \Big\{\emptyset, \{\omega_{FC}\}, \{\omega_{FC}\}^{\complement}, \{\omega_{CF}\}, \{\omega_{CF}\}^{\complement}, \{\omega_{CR}\}, \{\omega_{CR}\}^{\complement}, \{\omega_{RC}\}, \{\omega_{RC}\}^{\complement},$$
$$\{\omega_{FC}, \omega_{RC}\}, \{\omega_{CF}, \omega_{CR}\}, \{\omega_{FC}, \omega_{CR}\}, \{\omega_{CF}, \omega_{RC}\}, \Omega_{NR}, \Omega_{NR}^{\complement}, \Omega\Big\}$$

Example (2)

The information structure $\mathcal{F}(0), \mathcal{F}(1), \mathcal{F}(2)$ can be denoted by a tree:

no info.
$\begin{pmatrix} \omega \in \Omega \\ \omega \notin \emptyset \end{pmatrix}$

- Fcs: NR
  $\begin{pmatrix} \omega \in \Omega_{NR}, \Omega \\ \omega \notin \Omega_{NR}^{\complement}, \emptyset \end{pmatrix}$
  - Cty: F, Mt: C
    $\begin{pmatrix} \omega \in \{\omega_{FC}\}, \Omega_{NR}, \cdots, \Omega \\ \omega \notin \{\omega_{FC}\}^{\complement}, \Omega_{NR}^{\complement}, \cdots, \emptyset \end{pmatrix}$
  - Cty: C, Mt: F
    $\begin{pmatrix} \omega \in \{\omega_{CF}\}, \Omega_{NR}, \cdots, \Omega \\ \omega \notin \{\omega_{CF}\}^{\complement}, \Omega_{NR}^{\complement}, \cdots, \emptyset \end{pmatrix}$
- Fcs: NF
  $\begin{pmatrix} \omega \in \Omega_{NF}, \Omega \\ \omega \notin \Omega_{NF}^{\complement}, \emptyset \end{pmatrix}$
  - Cty: C, Mt: R
    $\begin{pmatrix} \omega \in \{\omega_{CR}\}, \Omega_{NF}, \cdots, \Omega \\ \omega \notin \{\omega_{CR}\}^{\complement}, \Omega_{NF}^{\complement}, \cdots, \emptyset \end{pmatrix}$
  - Cty: R, Mt: C
    $\begin{pmatrix} \omega \in \{\omega_{RC}\}, \Omega_{NF}, \cdots, \Omega \\ \omega \notin \{\omega_{RC}\}^{\complement}, \Omega_{NF}^{\complement}, \cdots, \emptyset \end{pmatrix}$

The Value for the TTP

Each buyer evaluates her value (reservation price / willingness-to-pay) for the TTP at each trading time, as an expectation of the exercise value conditional to the information at t.

We define the value of buyer i at t on ω by the conditional expectation of the exercise value given $\mathcal{F}(t)$:

$$v_i(t,\omega) = \mathbb{E}\left[\tilde{V}_i \middle| \mathcal{F}(t)\right](\omega),$$

where

$\mathbb{E}\left[\cdot \middle| \mathcal{F}(t)\right]$: conditional expectation given $\mathcal{F}(t)$.

Stochastic Process and its Measurability

$v_i(t, \omega)$ is a $\mathcal{F}(t)$-adaptive stochastic process:

- For given $t \in \mathcal{T}$, $\tilde{v}_i(t) = \{v_i(t, \omega) : \omega \in \Omega\}$ is a $\mathcal{F}(t)$-measurable[4] random variable.
- For given $\omega \in \mathcal{T}$, $\{v_i(t, \omega) : t \in \mathcal{T}\}$ is a (deterministic) sequence of the values.

A $\mathcal{F}(0)$-measurable variables, e.g. $v_i(0, \omega)$, is regarded as a constant (i.e. $\mathbb{E}[\tilde{v}_i(0)] = v_i(0)$ for any ω), since we can observe its outcome at the initial time.

Definition (Measurability)

A random variable $X : \Omega \to \mathcal{X}$ is measurable with respect to information $\mathcal{B}$ means that it is uncertain if the available information is smaller than $\mathcal{B}$, while it becomes certain if $\mathcal{B}$ or more information is available (i.e. the outcome of $\tilde{X}$ is included in $\mathcal{B}$).

[4] More rigorously, a random variable $X : \Omega \to \mathcal{X}$ is $\mathcal{F}$-measurable if its inversion set $X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\}$ for any subset $A \subset \mathcal{X}$ belongs to $\mathcal{F}$.

Outline

Feasible Assignments

$\mu(t)$: the number of TTPs traded at t (a given constant sequence)

$\boldsymbol{y}(t,\omega) := \{y_i(t,\omega) : i \in \mathcal{I}\}$ the assignment at t, where $y_i(t,\omega) = 1$ if a TTP is assigned to buyer i at t on ω; otherwise, $y_i(t,\omega) = 0$.

We say that an assignment $\boldsymbol{y} := \{\boldsymbol{y}(t,\omega) : (t,\omega) \in \mathcal{T} \times \Omega\}$ is feasible if, for any $\omega \in \Omega$,

- $\sum_{t\in\mathcal{T}} y_i(t,\omega) \leq 1$: Each customer is assigned at most one TTP;
- $\sum_{i\in\mathcal{I}} y_i(t,\omega) \leq \mu(t) \quad \forall t \in \mathcal{T}$: At most $\mu(t)$ TTPs can be assigned at t;
- $y_i(t,\omega) \in \{0,1\} \quad \forall (i,t) \in \mathcal{I} \times \mathcal{T}$.

$y_i(t,\omega)$ is also $\mathcal{F}(t)$-adaptive stochastic process.

Example ($\mu(0) = \mu(1) = \mu(2) = 1$)

$\mu(0) = 1$	$\mu(1) = 1$	$\mu(2) = 1$

- no info. $\boldsymbol{y}(0) = (0, 0, 1)$
  - Fcs: NR $\boldsymbol{y}(1,\omega) = (1, 0, 0)$
    - Cty: F, Mt: C $\boldsymbol{y}(2,\omega) = (0, 1, 0)$
    - Cty: C, Mt: F $\boldsymbol{y}(2,\omega) = (0, 1, 0)$
  - Fcs: NF $\boldsymbol{y}(1,\omega) = (0, 0, 0)$
    - Cty: C, Mt: R $\boldsymbol{y}(2,\omega) = (1, 0, 0)$
    - Cty: R, Mt: C $\boldsymbol{y}(2,\omega) = (0, 1, 0)$

- $\sum_{t\in\mathcal{T}} y_i(t,\omega) \le 1 \quad \forall (i,\omega) \in \mathcal{I}\times\Omega$
- $\sum_{i\in\mathcal{I}} y_i(t,\omega) \le \mu(t) \quad \forall t \in \mathcal{T}$

Outline

Optimal TTP Assignment Problem

Given $\boldsymbol{\mu} := \{\mu(t) : t \in \mathcal{T}\}$, the optimal TTP assignment problem is formulated as

$$\text{[P0]} \quad \max_{\boldsymbol{y}} \mathbb{E}\left[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{I}} \tilde{v}_i(t)\tilde{y}_i(t)\right]$$

$$\text{s.t.} \sum_{t\in\mathcal{T}} y_i(t,\omega) \le 1 \qquad \forall \omega \in \Omega,$$

$$\sum_{i\in\mathcal{I}} y_i(t,\omega) \le \mu(t) \quad \forall (t,\omega) \in \mathcal{T}\times\Omega$$

$$y_i(t,\omega) \in \{0,1\} \qquad \forall (i,t,\omega) \in \mathcal{I}\times\mathcal{T}\times\Omega$$

Example

$\mu(0) = 1$ | $\mu(1) = 1$ | $\mu(2) = 1$

no info.
$$\boldsymbol{v}(0) = (4, 4, 2)$$
$$\boldsymbol{y}(0) = (0, 0, 1)$$

Fcs: NR
$$\boldsymbol{v}(1,\omega) = (7, 7, 2)$$
$$\boldsymbol{y}(1,\omega) = (1, 0, 0)$$

Cty: F, Mt: C
$$\boldsymbol{v}(2,\omega) = (8, 6, 2)$$
$$\boldsymbol{y}(2,\omega) = (0, 1, 0)$$

Cty: C, Mt: F
$$\boldsymbol{v}(2,\omega) = (6, 8, 2)$$
$$\boldsymbol{y}(2,\omega) = (0, 1, 0)$$

Fcs: NF
$$\boldsymbol{v}(1,\omega) = (1, 1, 2)$$
$$\boldsymbol{y}(1,\omega) = (0, 0, 0)$$

Cty: C, Mt: R
$$\boldsymbol{v}(2,\omega) = (6, -4, 2)$$
$$\boldsymbol{y}(2,\omega) = (1, 0, 0)$$

Cty: R, Mt: C
$$\boldsymbol{v}(2,\omega) = (-4, 6, 2)$$
$$\boldsymbol{y}(2,\omega) = (0, 1, 0)$$

$$\mathbb{E}[\textstyle\sum_i v_i(0) y_i(0)] = 2$$

$$\mathbb{E}[\textstyle\sum_i \tilde{v}_i(1) \tilde{y}_i(1)] = \frac{7+0}{2} = 3.5$$

$$\mathbb{E}[\textstyle\sum_i \tilde{v}_i(2) \tilde{y}_i(2)] = \frac{6+8+6+6}{4} = 6.5$$

LP-relaxation

Since [P0] is an assignment problem with unit-demand, we believe[5] that it can be reduced to the following LP (*linear programming*):

$$
\begin{aligned}
\text{[P1]} \quad & \max_{\boldsymbol{y}} \mathbb{E}\left[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{I}} \tilde{v}_i(t)\tilde{y}_i(t)\right] \\
\text{s.t.} \quad & \sum_{t\in\mathcal{T}} y_i(t,\omega) \leq 1 && \forall \omega \in \Omega, \\
& \sum_{i\in\mathcal{I}} y_i(t,\omega) \leq \mu(t) && \forall (t,\omega) \in \mathcal{T}\times\Omega \\
& y_i(t,\omega) \geq 0 && \forall (i,t,\omega) \in \mathcal{I}\times\mathcal{T}\times\Omega
\end{aligned}
$$

[5] A rigorous proof remains future work.

Outline

Purpose of This Section

This section shows that the social optimal TTP assignment, i.e. the optimal solution for [P1], can be achieved as a Bayesian-Nash equilibrium of a sequence of competitive markets.

Outline

The Remaining Purchase Opportunity

Before obtain the optimality condition, we introduce a redundant $\mathcal{F}(t)$-adaptive stochastic process $z_i(t,\omega)$ that represents the remaining purchase opportunity of buyer i at t on ω.

From the definition, $\boldsymbol{z}(t,\omega) := \{z_i(t,\omega) : i \in \mathcal{I}\}$ must satisfy

$$
\begin{aligned}
y_i(0,\omega) + z_i(0,\omega) &\leq 1, \\
y_i(t,\omega) + z_i(t,\omega) &\leq z_i(t-1,\omega) \quad &\forall t \in \mathcal{T} \setminus \{0\}, \\
z_i(t,\omega) &\geq 0, \quad &\forall t \in \mathcal{T} \setminus \{\mathrm{T}\}.
\end{aligned}
$$

for any $(i,\omega) \in \mathcal{I} \times \Omega$.
For the notational simplicity, we introduce constants $z_i(-1,\omega) = 1$ and let $z_i(\mathrm{T},\omega) = 0$ for all $\omega \in \Omega$:

$$
\begin{aligned}
y_i(t,\omega) + z_i(t,\omega) &\leq z_i(t-1,\omega) \quad &\forall t \in \mathcal{T}, \\
z_i(t,\omega) &\geq 0, \quad &\forall t \in \mathcal{T}
\end{aligned}
$$

Reformulation of the TTP Assignment Problem

By using $\boldsymbol{z} := \{z(t,\omega) : (t,\omega) \in \mathcal{T} \times \Omega\}$, [P1] can be rewritten as

$$[\text{P}] \quad \max_{\boldsymbol{y},\boldsymbol{z}} \mathbb{E}\left[\sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{I}} \tilde{v}_i(t)\tilde{y}_i(t)\right]$$

$$\begin{aligned} \text{s.t. } & y_i(t,\omega) + z_i(t,\omega) \le z_i(t-1,\omega) && \forall (i,t,\omega) \\ & \sum_{i\in\mathcal{I}} y_i(t,\omega) \le \mu(t) && \forall (t,\omega) \\ & y_i(t,\omega), z_i(t,\omega) \ge 0 && \forall (i,t,\omega) \end{aligned}$$

Outline

The Value Function

For any $t < \mathrm{T}$, define the remaining duration $\mathcal{T}(t) := \{t, t+1, \cdots, \mathrm{T}\}$. Suppose that, at $t < \mathrm{T}$ with information $\mathcal{F}(t)$, the remaining purchase opportunity is $z_i(t-1,\omega) = z_i^-$ for $i \in \mathcal{I}$. Let the set of feasible remaining assignment be

$$\mathcal{Y}(t, \boldsymbol{z}^-) := \left\{ \begin{array}{ll} \Big(\tilde{\boldsymbol{y}}(s), \tilde{\boldsymbol{z}}(s)\Big) : s \in \mathcal{T}(t) & \text{such that} \\ \sum_{s \in \mathcal{T}(t)} y_i(s,\omega) \leq z_i^- & \forall (i,\omega) \in \mathcal{I} \times \Omega, \\ \sum_{i \in \mathcal{I}} y_i(s,\omega) \leq \mu(s) & \forall (i,s) \in \mathcal{I} \times \mathcal{T}(t) \\ y_i(s,\omega) \geq 0 & \forall (i,s,\omega) \in \mathcal{I} \times \mathcal{T}(t) \times \Omega \end{array} \right\}$$

We then define the value function of [P] at this situation as

$$\mathcal{J}\Big(t, \boldsymbol{z}^-\Big) := \max_{\{\tilde{\boldsymbol{y}}(s), \tilde{\boldsymbol{z}}(s)\} \in \mathcal{Y}(t, \boldsymbol{z}^-)} \sum_{s \in \mathcal{T}(t)} \sum_{i \in \mathcal{I}} \mathbb{E}\left[\tilde{v}_i(s) \tilde{y}_i(s) \middle| \mathcal{F}(t)\right].$$

Dynamic Programming

By applying DP (*dynamic programming*) principle, the TTP assignment problem [P] can be decomposed with respect to time: For any $t < 0$, the value function follows the recursive form:

$$\mathcal{J}(t, z^-) = \max_{\boldsymbol{y}(t),\boldsymbol{z}(t)} \sum_{i \in \mathcal{I}} v_i(t) y_i(t) + \mathbb{E}\left[\mathcal{J}\Big(t+1, \boldsymbol{z}(t)\Big)\Big|\mathcal{F}(t)\right],$$

$$\text{s.t. } y_i(t) + z_i(t) \le z_i^- \qquad \forall i \in \mathcal{I}, \qquad [\psi_i(t)]$$

$$\textstyle\sum_{i \in \mathcal{I}} y_i(t) \le \mu(t), \qquad [p(t)]$$

$$y_i(t), z_i(t) \ge 0, \forall i \in \mathcal{I}.$$

where the variable in each bracket is the corresponding Lagrange multipliers.

The variables at t, i.e. $v_i(t)$, $y_i(t)$, $z_i(t)$, $\psi_i(t)$ and $p(t)$ are $\mathcal{F}(t)$-measurable (i.e. certain with the information $\mathcal{F}(t)$).

The Value function at the Exercise Time

At the exercise time $t = \mathrm{T}$, the value function follows

$$\mathcal{J}(\mathrm{T}, \boldsymbol{z}^-) = \max_{\boldsymbol{y}(\mathrm{T})} \sum_{i \in \mathcal{I}} v_i(\mathrm{T}) y_i(\mathrm{T}),$$

$$\begin{aligned} \text{s.t. } & y_i(\mathrm{T}) \le z_i^- \qquad \forall i \in \mathcal{I}, \qquad [\psi_i(\mathrm{T})] \\ & \textstyle\sum_{i \in \mathcal{I}} y_i(\mathrm{T}) \le \mu(\mathrm{T}), \qquad [p(\mathrm{T})] \\ & y_i(\mathrm{T}) \ge 0 \qquad \forall i \in \mathcal{I} \end{aligned}$$

The Optimality Condition (1)

The optimality condition at $t < 0$ can be obtained as

$$
\begin{aligned}
&0 \leq y_i(t) \perp \{\psi_i(t) - v_i(t) + p(t)\} \geq 0 && \forall i \in \mathcal{I} \\
&0 \leq z_i(t) \perp \left\{\psi_i(t) - \mathbb{E}\left[\frac{\partial \mathcal{J}(t+1, z^-)}{\partial z_i(t)} \middle| \mathcal{F}(t)\right]\right\} \geq 0 && \forall i \in \mathcal{I} \\
&0 \leq \psi_i(t) \perp \left\{z_i^- - y_i(t) - z_i(t)\right\} \geq 0 && \forall i \in \mathcal{I} \\
&0 \leq p(t) \perp \{\mu(t) - \textstyle\sum_{i \in \mathcal{I}} y_i(t)\} \geq 0
\end{aligned}
$$

Since the Lagrange multiplier is equivalent to the corresponding shadow price, we have $\frac{\partial \mathcal{J}(t, z^-)}{\partial z_i^-} = \psi_i(t)$. Utilizing this, the second complementarity condition reduces to

$$
0 \leq z_i(t) \perp \left\{\psi_i(t) - \mathbb{E}\left[\tilde{\psi}_i(t+1) \middle| \mathcal{F}(t)\right]\right\} \geq 0
$$

The Optimality Condition (2)

The optimality condition of [P] can be summarized as:

- At any pre-sale market $t < \mathrm{T}$ with information $\mathcal{F}(t)$,

$$
\begin{aligned}
0 \le y_i(t) \perp \{\psi_i(t) - v_i(t) + p(t)\} \ge 0 \quad & \forall i \in \mathcal{I} \\
0 \le z_i(t) \perp \left\{\psi_i(t) - \mathbb{E}\left[\tilde{\psi}_i(t+1) \middle| \mathcal{F}(t)\right]\right\} \ge 0 \quad & \forall i \in \mathcal{I} \\
0 \le \psi_i(t) \perp \left\{z_i^- - y_i(t) - z_i(t)\right\} \ge 0 \quad & \forall i \in \mathcal{I} \\
0 \le p(t) \perp \{\mu(t) - \textstyle\sum_{i \in \mathcal{I}} y_i(t)\} \ge 0 \quad &
\end{aligned}
$$

- At the exercise time $t = \mathrm{T}$ with information $\mathcal{F}(\mathrm{T})$,

$$
\begin{aligned}
0 \le y_i(\mathrm{T}) \perp \{\psi_i(\mathrm{T}) - v_i(\mathrm{T}) + p(\mathrm{T})\} \ge 0 \quad & \forall i \in \mathcal{I} \\
0 \le \psi_i(\mathrm{T}) \perp \left\{z_i^- - y_i(\mathrm{T})\right\} \ge 0 \quad & \forall i \in \mathcal{I} \\
0 \le p(\mathrm{T}) \perp \{\mu(\mathrm{T}) - \textstyle\sum_{i \in \mathcal{I}} y_i(\mathrm{T})\} \ge 0 \quad &
\end{aligned}
$$

Outline

Implications of Bayesian-Nash Equilibrium (1)

The equilibrium implication can be dervied from:

1. the each buyer's demand of the TPP, $y_i(t, \omega)$, is generated from her self-interested maximization behavior of the expected payoff;
2. the price of the TPP, $p(t, \omega)$, is determined to match the supply $\mu(t)$ and the demand $\sum_{i \in \mathcal{I}} y_i(t, \omega)$.

for any market t on ω. First, let us give the following interpreta-

Implications of Bayesian-Nash Equilibrium (2)

tions for the Lagrange multipliers at t with information $\mathcal{F}(t)$:

$p(t)$: the price of TTP at t trade.

$\psi_i(t)$ the maximum expected payoff of buyer i from the remaining trades $\mathcal{F}(t)$, conditional to $\mathcal{F}(t)$. In other words, $\psi_i(t)$ is a value of buyer i for remaining in the market on and after t.

Implications of Bayesian-Nash Equilibrium (3)

- The demand is generated from the buyers' maximization

  This implication stems from the complementarity conditions at t with $\mathcal{F}(t)$: [D],

$$0 \leq y_i(t) \perp \{\psi_i(t) - v_i(t) + p(t)\} \geq 0 \qquad \forall i \in \mathcal{I}$$

$$0 \leq z_i(t) \perp \left\{\psi_i(t) - \mathbb{E}\left[\frac{\partial \mathcal{J}(t+1, z^-)}{\partial z_i(t)} \middle| \mathcal{F}(t)\right]\right\} \geq 0 \qquad \forall i \in \mathcal{I}$$

$$0 \leq \psi_i(t) \perp \left\{z_i^- - y_i(t) - z_i(t)\right\} \geq 0 \qquad \forall i \in \mathcal{I}$$

  Each can be interpreted as

  $y_i(t,\omega) > 0 \rightarrow \psi_i(t) = v_i(t) - p(t)$ and $z_i^- > 0$: buyer i immediately purchases the TTP and yields the payoff $v_i(t,\omega) - p(t,\omega)$ (if possible), only if it is possible and it maximizes her own expected payoff.

Implications of Bayesian-Nash Equilibrium (4)

$z_i(t) > 0 \rightarrow \psi_i(t) = \mathbb{E}\left[\tilde{\psi}_i(t+1) \middle| \mathcal{F}(t)\right]$: buyer i carries over her remaining purchase opportunity (i.e. she postpone the purchase) and $\mathbb{E}\left[\tilde{\psi}_i(t+1) \middle| \mathcal{F}(t)\right]$ only if it exists and maximizes her own expected payoff. $\mathbb{E}\left[\tilde{\psi}_i(t+1) \middle| \mathcal{F}(t)\right]$ is thus can be interpreted as the option value for the postponing.

$y_i(t) = z_i(t) = 0 \rightarrow \psi_i(t, \omega) = 0$: buyer i quits from the market only if no positive payoff is expected.

This can be summarized as: The buyer i's TPP demand, $y_i(t)$, is generated from her self-interested maximization of the expected payoff conditional to the available information $\mathcal{F}(t)$ and her remaining purchase opportunity $z_i(t - 1o)$.

Implications of Bayesian-Nash Equilibrium (5)

- The price of the TTP matches the demand and supply:

  The remaining complementarity conditions at t with information $\mathcal{F}(t)$:

  $$0 \le p(t) \perp \left\{ \mu(t) - \sum_{i \in \mathcal{I}} y_i(t) \right\} \ge 0$$

  can be interpreted as the standard market clearing conditions at t.

Implications of Bayesian-Nash Equilibrium (6)

Efficiency of the Bayesian-Nash Equilibrium

The social efficient assignment can be achieved as a Bayesian-Nash equilibrium of self-interested buyers in a sequence of competitive TTP markets, where the buyers have the identical information structure and $\{\mathcal{F}(t) : t \in \mathcal{T}\}$ and belief on $(\Omega, \mathcal{F})$.

Outline

Summary

- We formulate an optimal assignment of the TTP with multiple purchase opportunity under dynamic uncertainty as a stochastic control problem.
- The optimality condition is obtained by applying the *Dynamic Programming* (DP) principle
- Our analyses reveals that the optimal assignment can be achieved as a Bayesian-Nash equilibrium of self-interested buyers in a sequence of competitive markets.

Future work

- To examine the existence of strategy-proof and individually rational mechanisms that achieves the social optimal assignment.
- To develop an efficient algorithm to find the social optimal allocation $\boldsymbol{\mu}(t) = \{\mu(t) : t \in \mathcal{T}\}$ with the capacity constraint $\sum_{t\in\mathcal{T}} \mu(t) \leq \mu$, where μ is the total capacity.