Matching With Minimal Priority Rights

Szilvia Pápai

Corvinus University

Concordia University

Priority – Based Matching Problems

Set of objects: $\boldsymbol{M}$ $(\boldsymbol{a} \in \boldsymbol{M})$

Set of agents: $\boldsymbol{N}$ $(\boldsymbol{i} \in \boldsymbol{N})$

Each agent is allocated at most one object based on priorities of the agents for objects.

Each object has a strict priority ranking of agents $(\boldsymbol{\pi_a})$

Each agent has a strict preference ranking of objects $(\boldsymbol{R_i})$

All objects are assumed to be goods.

Set of objects: $M = \{a, b, c, d, e, f, g\}$

Set of agents: $N = \{1, 2, 3, 4, 5, 6, 7\}$

Priorities π

a	b	c	d	e	f	g
1	6	4	4	5	4	6
4	5	5	5	6	5	5
5	4	6	1	4	6	4
6	1	1	6	1	1	1
3	3	2	2	3	7	2
2	7	3	3	7	2	7
7	2	7	7	2	3	3

Preferences R

1	2	3	4	5	6	7
d	f	e	e	a	e	d
a	b	f	g	b	a	a
e	c	c	c	c	f	c
b	g	d	b	d	b	b
c	a	b	f	e	g	g
g	d	a	d	f	d	e
f	e	g	a	g	c	f

What do the priorities mean?

There are two distinct interpretations of the priorities, corresponding to two well-known matching rules.

What do the priorities mean?

There are two distinct interpretations of the priorities, corresponding to two well-known matching rules.

1. **SIMPLE PRIORITIES**

Deferred Acceptance rule **(DA)** – *agent-optimal stable matching*

(Gale-Shapley, 1962)

What do the priorities mean?

There are two distinct interpretations of the priorities, corresponding to two well-known matching rules.

1. **SIMPLE PRIORITIES**

Deferred Acceptance rule **(DA)** – *agent-optimal stable matching*

(Gale-Shapley, 1962)

2. **TRADE PRIORITIES**

Top Trading Cycle rule **(TTC)** *with inheritance*

(Shapely-Scarf, 1974 →Gale; Pápai, 2000)

SIMPLE PRIORITIES: *Deferred Acceptance rule* (DA)

Priorities π

a	b	c	d	e	f	g
1	6	4	4	5	4	6
4	5	5	5	6	5	5
5	4	6	1	4	6	4
6	1	1	6	1	1	1
3	3	2	2	3	7	2
2	7	3	3	7	2	7
7	2	7	7	2	3	3

Preferences R

1	2	3	4	5	6	7
d	f	e	e	a	e	d
a	b	f	g	b	a	a
e	c	c	c	c	f	c
b	g	d	b	d	b	b
c	a	b	f	e	g	g
g	d	a	d	f	d	e
f	e	g	a	g	c	f

Agents "propose" to objects iteratively.

TRADE PRIORITIES: *Top Trading Cycle rule* (TTC)

Priorities π

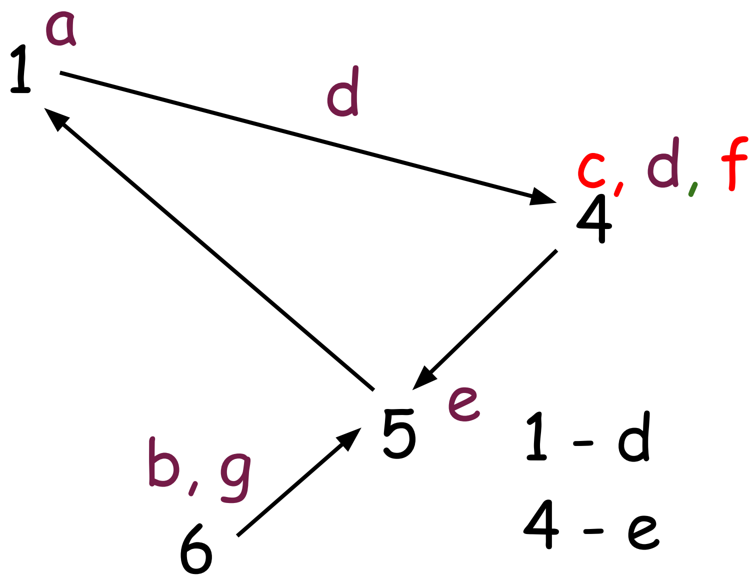
a	b	c	d	e	f	g
1	6	4	4	5	4	6
4	5	5	5	6	5	5
5	4	6	1	4	6	4
6	1	1	6	1	1	1
3	3	2	2	3	7	2
2	7	3	3	7	2	7
7	2	7	7	2	3	3

Preferences R

1	2	3	4	5	6	7
d	f	e	e	a	e	d
a	b	f	g	b	a	a
e	c	c	c	c	f	c
b	g	d	b	d	b	b
c	a	b	f	e	g	g
g	d	a	d	f	d	e
f	e	g	a	g	c	f

Agents trade objects iteratively in top trading cycles, with inheritance of the objects.

Round 1

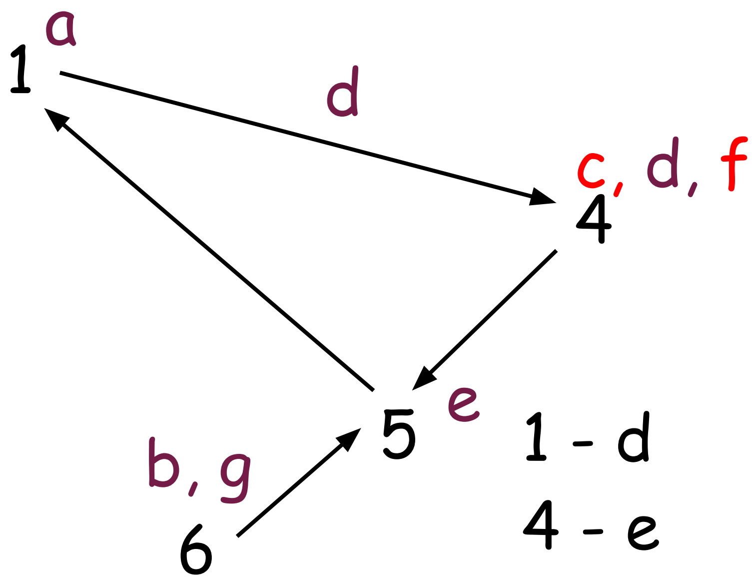


1 - d

4 - e

5 - a

Round 1



1 - d

4 - e

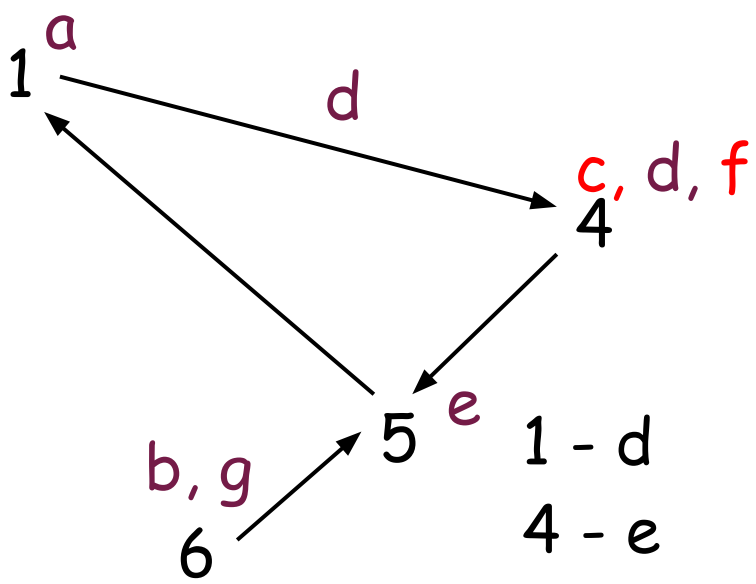
5 - a

Round 2

b, g, c, f

6

Round 1

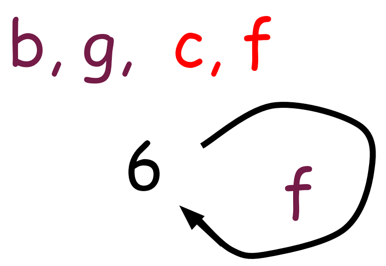


1 - d

4 - e

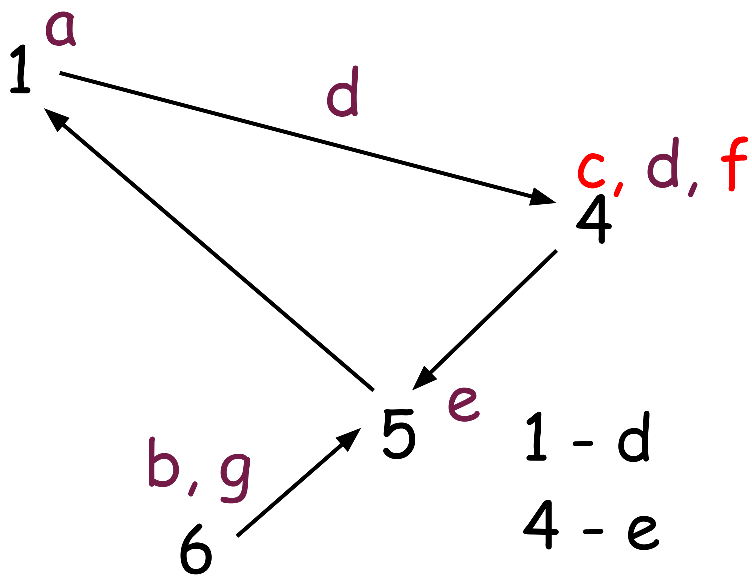
5 - a

Round 2



6 - f

Round 1

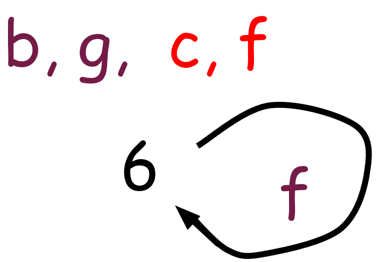


1 - d

4 - e

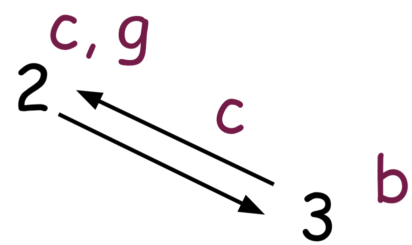
5 - a

Round 2



6 - f

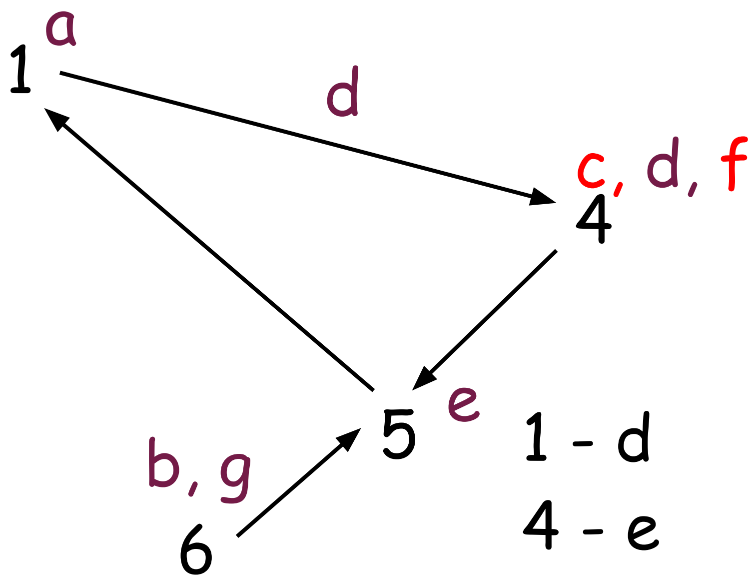
Round 3



2 - b

3 - c

Round 1

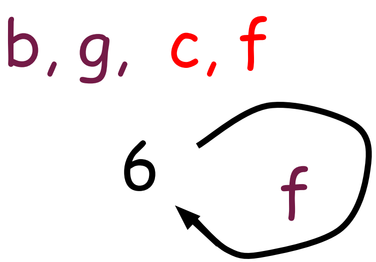


1 - d

4 - e

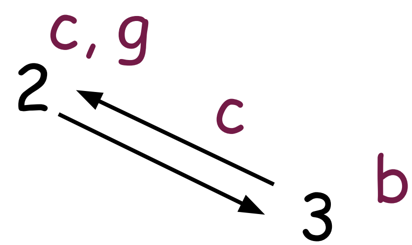
5 - a

Round 2



6 - f

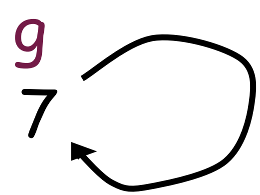
Round 3



2 - b

3 - c

Round 4



7 - g

Stability/Fairness

A matching rule $\boldsymbol{f}$ *is* *stable* *if, for all preference profiles* $\boldsymbol{R}$*, there are no agents* $\boldsymbol{i, j} \in \boldsymbol{N}$ *and object* $\boldsymbol{a} \in \boldsymbol{M}$ *such that*

- $\boldsymbol{a\, P_i\, f_i(R)}$
- $\boldsymbol{i\, \pi_a\, j}$
- $\boldsymbol{f_j(R) = a}$

Stability/Fairness

A matching rule $\boldsymbol{f}$ *is* *stable* *if, for all preference profiles* $\boldsymbol{R}$*, there are no agents* $\boldsymbol{i, j} \in \boldsymbol{N}$ *and object* $\boldsymbol{a} \in \boldsymbol{M}$ *such that*

- $\boldsymbol{a \, P_i \, f_i(R)}$
- $\boldsymbol{i \, \pi_a \, j}$
- $\boldsymbol{f_j(R) = a}$

The DA rule is stable. *(Gale-Shapley, 1962)*

The TTC rule is not stable.

Efficiency

A matching rule $\boldsymbol{f}$ *is* *<u>efficient</u>* *if, for all preference profiles* $\boldsymbol{R}$*,* $\boldsymbol{f}(\boldsymbol{R})$ *is not Pareto dominated.*

Efficiency

A matching rule $\boldsymbol{f}$ *is* *efficient* *if, for all preference profiles* $\boldsymbol{R}$*,* $\boldsymbol{f}(\boldsymbol{R})$ *is not Pareto dominated.*

The DA rule is not efficient.

The TTC rule is efficient.

Strategyproofness

A matching rule f *is* <u>*Strategyproof*</u> *if, for all preference profiles* $\boldsymbol{R}$*, all agents* $\boldsymbol{i} \in \boldsymbol{N}$*, and all* $\boldsymbol{R}'_i$*,*

$$\boldsymbol{f}_i(\boldsymbol{R})\ \boldsymbol{R}_i\ \boldsymbol{f}_i(\boldsymbol{R}'_i, \boldsymbol{R}_{-i}).$$

Strategyproofness

A matching rule f is Strategyproof if, for all preference profiles R, all agents $i \in N$, and all R'_i,

$$f_i(R)\ R_i\ f_i(R'_i, R_{-i}).$$

Both the DA and the TTC rules are Strategyproof.

(Dubins and Freedman, 1981; Roth, 1982)

Stability and efficiency cannot be reconciled for an arbitrary priority table.

With restricted priorities: stability and efficiency can be satisfied simultaneously only if the priority table is acyclic.

Ergin (2002)

Kesten (2006)

If the priority table is acyclic then the DA and TTC rules yield the same matching for every preference profile.

Questions posed in this paper:

- How to combine some of the nice features of the DA and TTC rules without restricting the priority table?

  Is there a class of combined rules that captures the tradeoff between stability and efficiency and allows greater flexibility than using either the DA or the TTC rule?

Questions posed in this paper:

- How to combine some of the nice features of the DA and TTC rules *without* restricting the priority table?

  Is there a class of combined rules that captures the *tradeoff between stability and efficiency* and allows greater flexibility than using either the DA or the TTC rule?

- Are there combined rules which are *strategyproof*?

Relaxing stability: Priority versus Trade Stability

Allow for trading priorities (just like in the TTC rule)

Stability rules out objections of the following type:

- **agent i prefers object a to the object he receives**
- **i has higher priority for object a than agent j does**
- **j receives object a**

Relaxing stability: Priority versus Trade Stability

Allow for trading priorities (just like in the TTC rule)

Stability rules out objections of the following type:

- **agent i prefers object a to the object he receives**
- **i has higher priority for object a than agent j does**
- **j receives object a**

This does not rule out the possibility that j traded object a with some agent l who has higher priority for a than i does.

Call agent i an intermediate agent for object a if this is the case.

TRADE STABILITY: i has a trade objection if, in addition,

- i is not an intermediate agent for object a (note: this depends on the matching)

How to define TRADE STABILITY formally..?

INTRODUCE: *Claim Procedure* → *in order to determine the intermediate agents*

INTRODUCE: *Claim Procedure* → *in order to determine the intermediate agents*

Claim Procedure at matching x:

Iterative procedure: in each round each agent points to the owner of x_i (note: the current owner may change!)

INTRODUCE: <u>Claim Procedure</u> → in order to determine the intermediate agents

<u>Claim Procedure at matching x:</u>

Iterative procedure: in each round each agent points to the owner of x_i (note: the current owner may change!)

- *Find all claim cycles in each round, remove agents and objects in claim cycles; note owners and recipients*

INTRODUCE: <u>Claim Procedure</u> → in order to determine the intermediate agents

<u>Claim Procedure at matching x:</u>

Iterative procedure: in each round each agent points to the owner of x_i (note: the current owner may change!)

- *Find all claim cycles in each round, remove agents and objects in claim cycles; note owners and recipients*
- *Allow for inheritance of the objects*

INTRODUCE: *Claim Procedure* → *in order to determine the intermediate agents*

Claim Procedure at matching x:

Iterative procedure: in each round each agent points to the owner of x_i (note: the current owner may change!)

- *Find all claim cycles in each round, remove agents and objects in claim cycles; note owners and recipients*
- *Allow for inheritance of the objects*
- *Repeat in the remaining market*

INTRODUCE: Claim Procedure → in order to determine the intermediate agents

Claim Procedure at matching x:

Iterative procedure: in each round each agent points to the owner of x_i (note: the current owner may change!)

- Find all claim cycles in each round, remove agents and objects in claim cycles; note owners and recipients
- Allow for inheritance of the objects
- Repeat in the remaining market
- Stop when all agents or all objects are removed

INTRODUCE: Claim Procedure → in order to determine the intermediate agents

Claim Procedure at matching x:

Iterative procedure: in each round each agent points to the owner of x_i (note: the current owner may change!)

- *Find all claim cycles in each round, remove agents and objects in claim cycles; note owners and recipients*
- *Allow for inheritance of the objects*
- *Repeat in the remaining market*
- *Stop when all agents or all objects are removed*

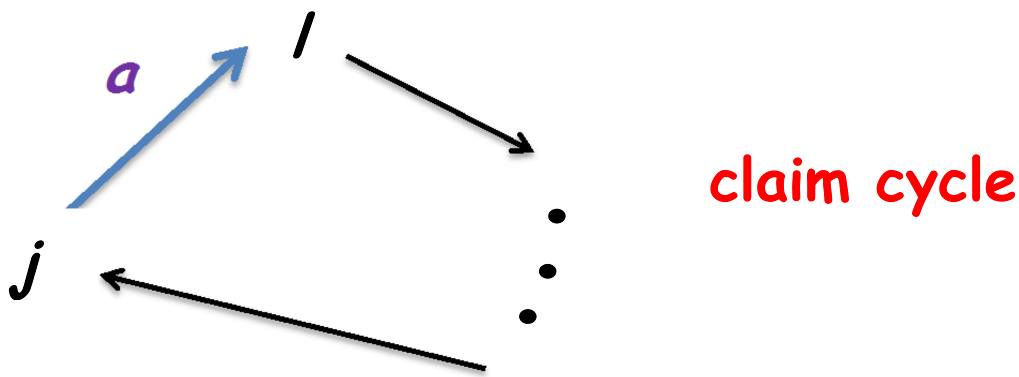
The procedure does not depend on the preferences, it depends only on matching x.

Intermediate agents

Agent i is an <u>*intermediate agent for object a at matching x*</u> *if $l\,\pi_a\, i\, \pi_a\, j$ for agents $l, j \in N$ such that there is a claim cycle in some round of the claim procedure at x in which l is the owner of a and j is the recipient of a (i.e., $x_j = a$.).*

For all objects $a \in M$, let $INT_a(x)$ denote the set of intermediate agents for a in the claim procedure at x.

π_a
l
i
j



Priority Stability (= standard stability)

Let agent i's priority claim set be

$$C_i^{Priority}(x) = \{a \in M : i\, \pi_a l_a\},$$

where $x_{l_a} = a$ *for all* $a \in M$.

Under simple priorities, each agent i is entitled to objects in $C_i^{Priority}(x)$.

Priority Stability (= standard stability)

Let agent i's *priority claim set* be

$$\boldsymbol{C}_i^{Priority}(\boldsymbol{x}) = \{\boldsymbol{a} \in \boldsymbol{M}: \boldsymbol{i}\, \boldsymbol{\pi}_a \boldsymbol{l}_a\},$$

where $\boldsymbol{x}_{l_a} = \boldsymbol{a}$ for all $\boldsymbol{a} \in \boldsymbol{M}$.

Under *simple priorities*, each agent i is entitled to objects in $\boldsymbol{C}_i^{Priority}(\boldsymbol{x})$.

Agent i has a *priority objection* to matching $\boldsymbol{x}$ at $\mathcal{R}$ if there exists $\boldsymbol{a} \in \boldsymbol{M}$ such that $\boldsymbol{a}\boldsymbol{P}_i\boldsymbol{x}_i$ and $\boldsymbol{a} \in \boldsymbol{C}_i^{Priority}(\boldsymbol{x})$.

Trade Stability (new weaker stability concept)

Let agent i's trade claim set be

$$C_i^{Trade}(x) = \{a \in M : i\, \pi_a l_a,\ \ i \notin INT_a(x)\},$$

where $x_{l_a} = a$ *for all* $a \in M$.

Under trade priorities, each agent i is entitled to objects in $C_i^{Trade}(x)$.

Compare to priority claim set:

$$C_i^{Priority}(x) = \{a \in M : i\, \pi_a l_a\}$$

Trade Stability (new weaker stability concept)

Let agent i's trade claim set be

$$\boldsymbol{C_i^{Trade}(x) = \{a \in M: i\, \pi_a l_a,\ \ i \notin INT_a(x)\}},$$

where $\boldsymbol{x_{l_a} = a}$ *for all* $\boldsymbol{a \in M}$.

Under trade priorities, each agent i is entitled to objects in $\boldsymbol{C_i^{Trade}(x)}$.

Agent i has a trade objection to matching x *at* $\mathcal{R}$ *if there exists* $\boldsymbol{a \in M}$ *such that* $\boldsymbol{aP_ix_i}$ *and* $\boldsymbol{a \in C_i^{Trade}(x)}$.

Trade claim set:

$$C_i^{Trade}(x) = \{a \in M: i\ \pi_a l_a,\ i \notin INT_a(x)\}$$

Priority claim set:

$$C_i^{Priority}(x) = \{a \in M: i\ \pi_a l_a\}$$

where $x_{l_a} = a$ *for all* $a \in M$.

Observe: $C_i^{Trade}(x) \subset C_i^{Priority}(x)$

Priority Stability $\Rightarrow$ *Trade Stability*

Individual versus Coalition Stability

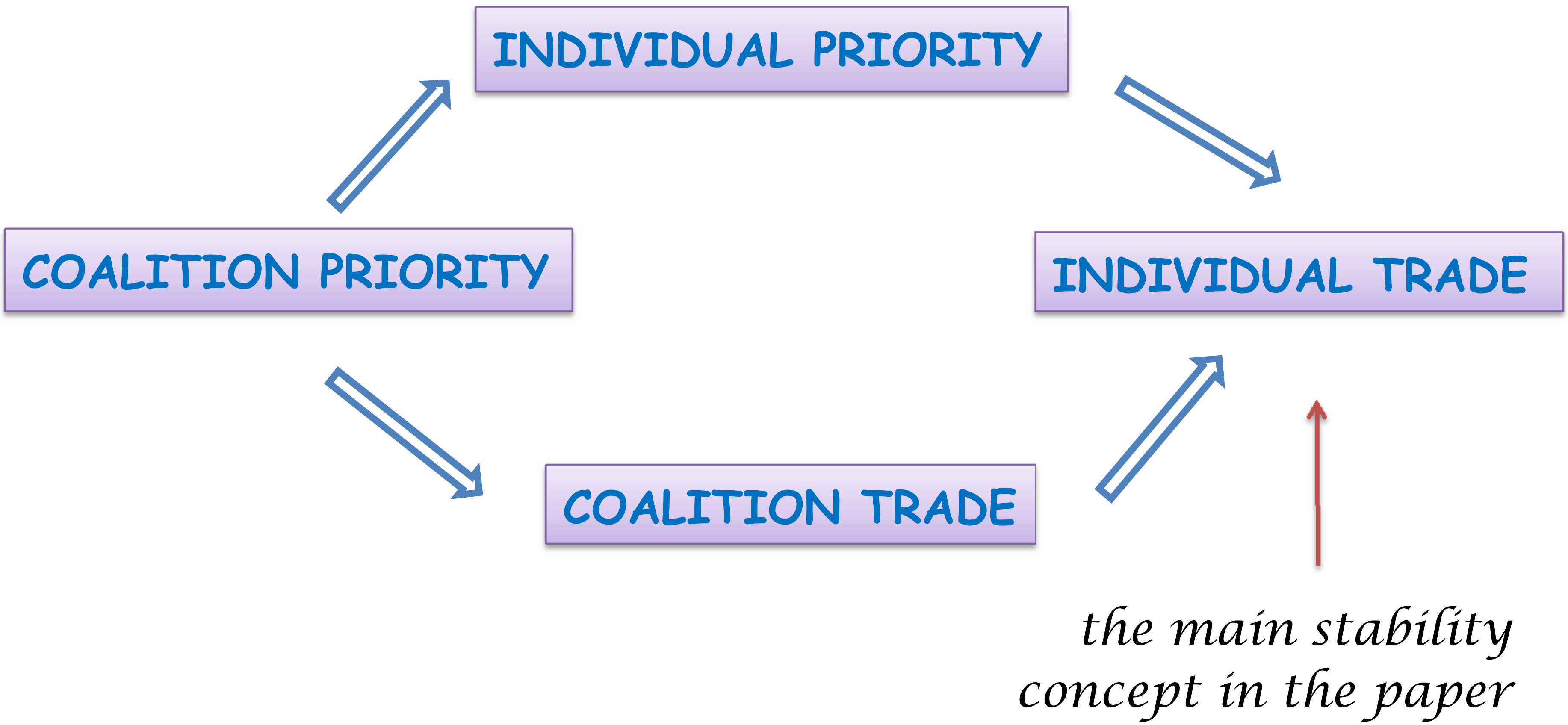
Both trade and individual stability only rule out objections by a single agent.

Now we will rule out objections by coalitions: members of a coalition can exchange objects from their claim sets.

This makes the stability concept more demanding:

Coalition Stability $\Rightarrow$ Individual Stability

Four stability concepts



Coalition Priority objection:

Coalition S has a coalition priority objection to matching x at $\mathcal{R}$ if there exist $\boldsymbol{y_S}$ and $\boldsymbol{z_S}$ such that

- $\boldsymbol{y_s}$ is a reallocation of $\boldsymbol{z_S}$
- $\boldsymbol{y_s}$ Pareto dominates $\boldsymbol{x_s}$
- for all $\boldsymbol{i} \in \boldsymbol{S},\ \boldsymbol{z_i} \in \boldsymbol{C_i^{Priority}}(\boldsymbol{x})$

Coalition Priority objection:

Coalition S has a coalition priority objection to matching x at $\mathcal{R}$ if there exist $\boldsymbol{y_S}$ and $\boldsymbol{z_S}$ such that

- $\boldsymbol{y_s}$ is a reallocation of $\boldsymbol{z_S}$
- $\boldsymbol{y_s}$ Pareto dominates $\boldsymbol{x_s}$
- for all $\boldsymbol{i} \in \boldsymbol{S}$, $\boldsymbol{z_i} \in \boldsymbol{C_i^{Priority}}(\boldsymbol{x})$

Coalition Trade objection:

Same as a coalition priority objection, except:

replace $\boldsymbol{C_i^{Priority}}(\boldsymbol{x})$ by $\boldsymbol{C_i^{Trade}}(\boldsymbol{x})$.

Four concepts of stable matching

Matching x is INDIVIDUAL / COALITION PRIORITY / TRADE *stable at R if*

there is no INDIVIDUAL / COALITION PRIORITY / TRADE *objection to x at R.*

Four concepts of stable matching rules

A matching rule f *is*

INDIVIDUAL	PRIORITY
COALITION	TRADE

stable if

for all preference profiles $\boldsymbol{R}$, $f(\boldsymbol{R})$ *is*

INDIVIDUAL	PRIORITY
COALITION	TRADE

stable at R.

	TRADE ⇦	PRIORITY
INDIVIDUAL ⇧	*Theorems 1, 2* (DA, TTC included)	(DA included)
COALITION	*Proposition 1* TTC only	*Proposition 2* ∅

Proposition 1

The only matching rule that satisfies Coalition Trade Stability is the TTC rule.

Proposition 1

The only matching rule that satisfies Coalition Trade Stability is the TTC rule.

Proposition 2

There is no matching rule that satisfies Coalition Priority Stability.

Proof idea for Proposition 2:

a	b	c
1	2	
3		
2		

1	2	3
b	a	a
a	b	

If $f_2(R) = a$:

Agent 3 has an individual priority objection, since $a \, P_3 f_3(R)$ and $a \in C_3^{Priority}\big(f(R)\big)$.

Proof idea for Proposition 2:

a	b	c
1	2	
3		
2		

1	2	3
b	a	a
a	b	

If $f_2(R) = a$:

Agent 3 has an individual priority objection, since $a \, P_3 f_3(R)$ and $a \in C_3^{Priority}(f(R))$.

If $f_2(R) \neq a$:

Given individual priority stability, $f_2(R) = b$ and thus $f_1(R) = a$.

Therefore, the coalition {1,2} has a coalition priority objection to $f(R)$.

Proof idea for Proposition 2:

a	b	c
1	2	
3		
2		

1	2	3
b	a	a
a	b	

If $f_2(R) = a$:

Agent 3 has an individual priority objection, since $a\ P_3 f_3(R)$ and $a \in C_3^{Priority}(f(R))$.

If $f_2(R) \neq a$:

Given individual priority stability, $f_2(R) = b$ and thus $f_1(R) = a$.

Therefore, the coalition {1,2} has a coalition priority objection to $f(R)$.

Observe: this result is related to Ergin (2002) and Kesten (2006).

The priority table contains a cycle.

Coalition Priority Stability implies both stability (directly) and efficiency (by Proposition 1) and thus can only be satisfied if the priority table is acyclic.

Deferred Trading Cycle rules (DTC)

A class of rules that combine the DA *and* TTC *rules*

Each DTC rule f^{τ} is associated with a *weak* priority table τ that is *consistent* with the fixed strict priority table π.

strict priorities

π: **primitive**

π_a	π_b	π_c	π_d
1	2	4	1
2	3	1	3
3	1	3	4
4	4	2	2

weak priorities

f^{τ}: **matching rule**

τ_a	τ_b	τ_c	τ_d
{1	2	4	1
2}	3	{1	{3
{3	1	3	4}
4}	4	2}	2

Consistent: "collapse" strict priority orderings of consecutive agents into equivalence/indifference classes

(note: the strict orderings still matter for inheritance!)

Deferred Trading Cycle rules (DTC) - *Description*

DA rounds:

Everyone points to the agent who currently owns his top-ranked object. Resolve conflicts based on the weak priority table τ. Repeat as many times as possible.
(Note: all matchings are temporary.)

Deferred Trading Cycle rules (DTC) - *Description*

After as many DA rounds as possible

TTC round:

Everyone points to the agent who currently owns his top-ranked object. If there are still conflicts but none of them can be resolved based on the weak priority table τ, then identify all top trading cycles, carry out the corresponding trades, and remove these agents with their objects from the market.

(Note: the matchings in top trading cycles are final.)

Deferred Trading Cycle rules (DTC) – *Description*

After as many DA rounds as possible and one TTC round, go back to DA rounds, and so on.

Go back to DA rounds after each TTC round, and keep repeating until either all agents or objects are matched and removed, or until there are no more conflicts remaining.

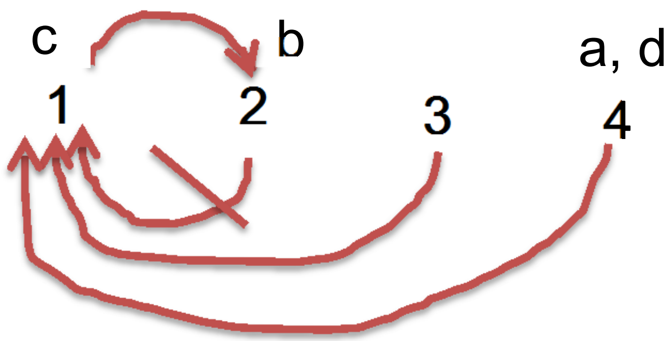
The final matching is given by the matchings in TTC rounds and by the final temporary matchings, if any remain.

Example: DTC *rule*

τ_a	τ_b	τ_c	τ_d
4	2	1	4
3	3	3	1
2	4	4	3
1	1	2	2

R_1	R_2	R_3	R_4
b	c	c	c
d	a	b	b
		d	a
		a	

Round 1: DA round

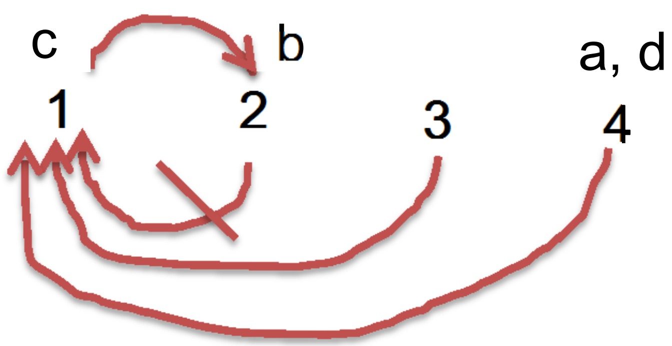


Example: DTC *rule*

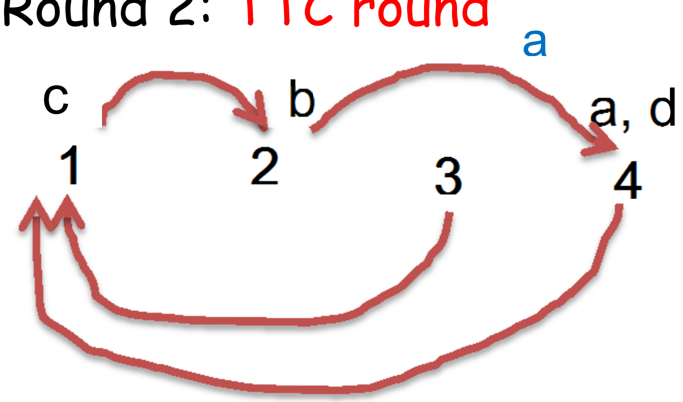
τ_a	τ_b	τ_c	τ_d
4	2	1	4
3	3	3	1
2	4	4	3
1	1	2	2

R_1	R_2	R_3	R_4
b	c	c	c
d	a	b	b
		d	a
		a	

Round 1: DA round



Round 2: TTC round



1 – b

2 – a

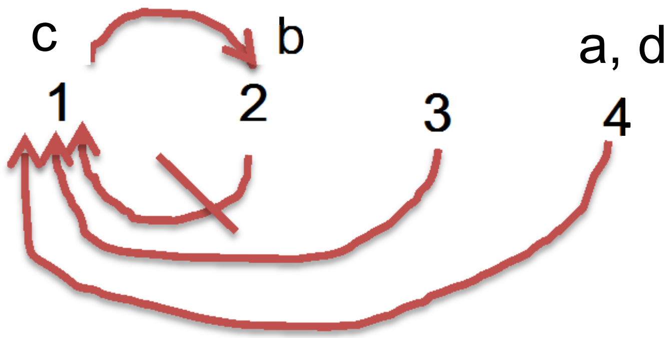
4 - c

Example: DTC *rule*

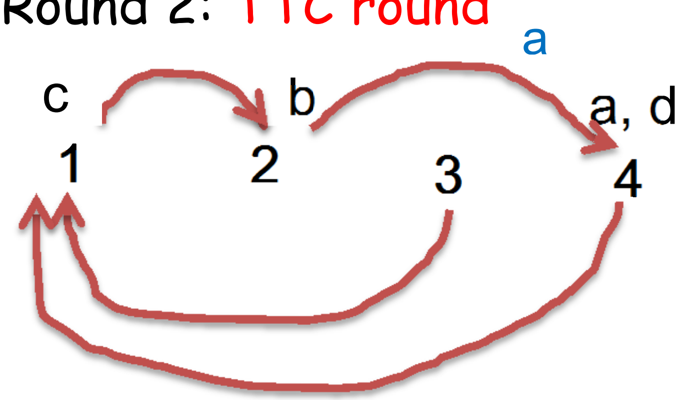
τ_a	τ_b	τ_c	τ_d
4	2	1	4
3	3	3	1
2	4	4	3
1	1	2	2

R_1	R_2	R_3	R_4
b	c	c	c
d	a	b	b
		d	a
		a	

Round 1: DA round



Round 2: TTC round

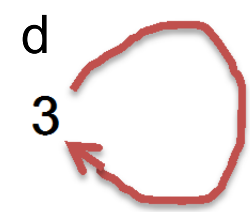


1 – b

2 – a

4 - c

Round 3



3 - d

DTC

R_1	R_2	R_3	R_4
b	c	c	c
d	a	b	b
		d	a
		a	

DA

R_1	R_2	R_3	R_4
b	c	c	c
d	a	b	b
		d	a
		a	

TTC

R_1	R_2	R_3	R_4
b	c	c	c
d	a	b	b
		d	a
		a	

Deferred Trading Cycle rules (DTC)

DTC rules combine properties of both the DA and TTC rules.

Deferred Trading Cycle rules (DTC)

DTC rules combine properties of both the DA and TTC rules.

The two extreme DTC rules are the DA and TTC rules.

Deferred Trading Cycle rules (DTC)

DTC rules combine properties of both the DA and TTC rules.

The two extreme DTC rules are the DA and TTC rules.

- if τ is all *strict*: DA rule

Deferred Trading Cycle rules (DTC)

DTC rules combine properties of both the DA and TTC rules.

The two extreme DTC rules are the DA and TTC rules.

- if τ is all ***strict***: DA rule
- if τ is all ***weak:*** TTC rule

Deferred Trading Cycle rules (DTC)

DTC rules combine properties of both the DA and TTC rules.

The two extreme DTC rules are the DA and TTC rules.

- if τ is all ***strict***: DA rule
- if τ is all ***weak:*** TTC rule

All DTC rules satisfy Individual Trade Stability.

Further properties of matching rules

Constrained Efficiency (←relaxing efficiency)

A matching rule $\boldsymbol{f}$ is <u>Constrained Efficient</u> if, for all preference profiles $\boldsymbol{R}$, if a matching $\boldsymbol{y}$ Pareto dominates $\boldsymbol{f}(\boldsymbol{R})$ then there exist agents $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{N}$ such that $\boldsymbol{j}\ \boldsymbol{\pi}_{\boldsymbol{y}_i}\ \boldsymbol{i}$, $\boldsymbol{y}_i\ \boldsymbol{P}_j \boldsymbol{y}_j$, and $\boldsymbol{y}_i \neq \boldsymbol{f}_i\ (\boldsymbol{R})$.

Further properties of matching rules

Constrained Efficiency (←relaxing efficiency)

A matching rule f *is* <u>*Constrained Efficient*</u> *if, for all preference profiles* $\boldsymbol{R}$*, if a matching* $\boldsymbol{y}$ *Pareto dominates* $\boldsymbol{f}(\boldsymbol{R})$ *then there exist agents* $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{N}$ *such that* $\boldsymbol{j}\, \boldsymbol{\pi}_{\boldsymbol{y}_i}\, \boldsymbol{i}$, $\boldsymbol{y}_i\, \boldsymbol{P}_j \boldsymbol{y}_j$, *and* $\boldsymbol{y}_i \neq \boldsymbol{f}_i\,(\boldsymbol{R})$.

Both the DA and the TTC rules are Constrained Efficient.

All DTC rules are Constrained Efficient.

Lower Invariance *(a mild incentive property)*

Let $\boldsymbol{R}_i^U(\boldsymbol{a})$ *be the preference ordering in* $\boldsymbol{R}_i$ *from the top-ranked object to object* $\boldsymbol{a}$.

Lower Invariance (a mild incentive property)

Let $\boldsymbol{R_i^U(a)}$ be the preference ordering in $\boldsymbol{R_i}$ from the top-ranked object to object $\boldsymbol{a}$.

A matching rule $\boldsymbol{f}$ satisfies Lower Invariance if, for all preference profiles $\boldsymbol{R}, \overline{\boldsymbol{R}}$ such that for all $\boldsymbol{i} \in \boldsymbol{N}$,

$$\boldsymbol{R_i^U\big(f_i(R)\big) = \overline{R}_i^U\big(f_i(R)\big)},$$

$\boldsymbol{f(R) = f(\overline{R})}$.

Lower Invariance (a mild incentive property)

Let $\boldsymbol{R}_i^U(\boldsymbol{a})$ *be the preference ordering in* $\mathbf{R}_i$ *from the top-ranked object to object* $\boldsymbol{a}$.

A matching rule $\boldsymbol{f}$ *satisfies* *Lower Invariance* *if, for all preference profiles* $\mathbf{R}, \overline{\mathbf{R}}$ *such that for all* $\boldsymbol{i} \in \boldsymbol{N}$,

$$\boldsymbol{R}_i^U\big(\boldsymbol{f}_i(\boldsymbol{R})\big) = \overline{\boldsymbol{R}}_i^U\big(\boldsymbol{f}_i(\boldsymbol{R})\big),$$

$\boldsymbol{f}(\boldsymbol{R}) = \boldsymbol{f}(\overline{\boldsymbol{R}})$.

Both the DA and the TTC rules satisfy Lower Invariance.

All DTC rules satisfy Lower Invariance.

Theorem 1:

Individual Trade Stability

Constrained Efficiency

Lower Invariance

$\Longleftrightarrow$ Deferred Trading Cycle Rules

Strategyproofness?

Not all Deferred Trading Cycle rules are Strategyproof.

If agent i, who is an intermediate agent for an object, may be "stalled" due to a tie ($\tilde{\tau}$) for another object then this agent can manipulate. We call this an *intermediate agent cycle.*

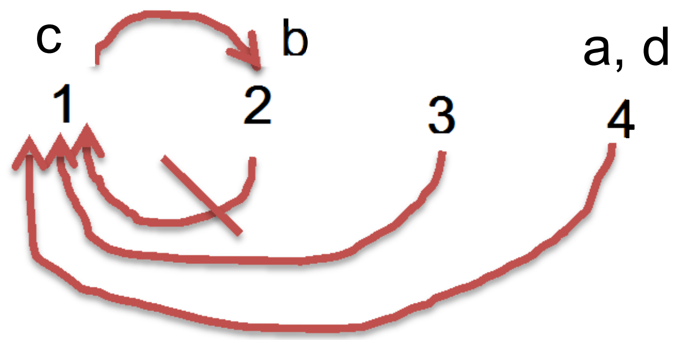
Recall example:

DTC *rule*

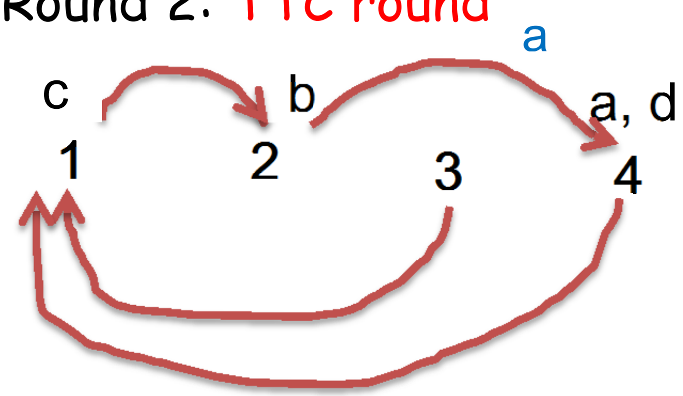
τ_a	τ_b	τ_c	τ_d
4	2	1	4
3	3	3	1
2	4	4	3
1	1	2	2

R_1	R_2	R_3	R_4
b	c	c	c
d	a	b	b
		d	a
		a	

Round 1: DA round



Round 2: TTC round

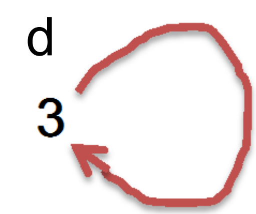


1 – b

2 – a

4 - c

Round 3



3 - d

Example: DTC *rules are not Strategyproof*

Agent 3 can manipulate:

3 can get object **b** instead of object **d** and bP_3d.

τ_a	τ_b	τ_c	τ_d
4	2	1	4
3	3	3	1
2	4	4	3
1	1	2	2

R_1	R_2	R'_3	R_4
b	c	b	c
d	a	d	b
		a	a
		c	

Example: DTC *rules are not Strategyproof*

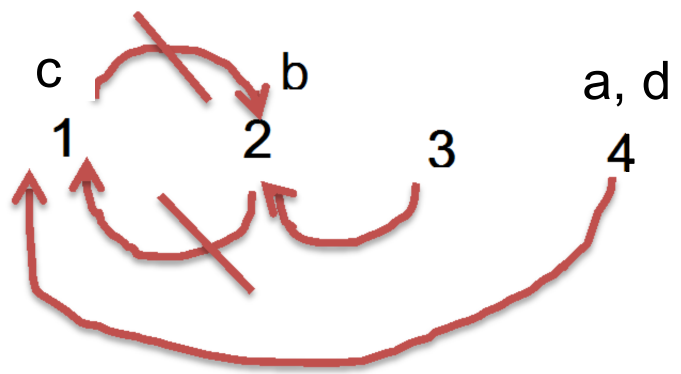
Agent 3 can manipulate:

3 can get object **b** instead of object **d** and bP_3d.

τ_a	τ_b	τ_c	τ_d
4	2	1	4
3	3	3	1
2	4	4	3
1	1	2	2

R_1	R_2	R'_3	R_4
b	c	b	c
d	a	d	b
		a	a
		c	

Round 1: DA round



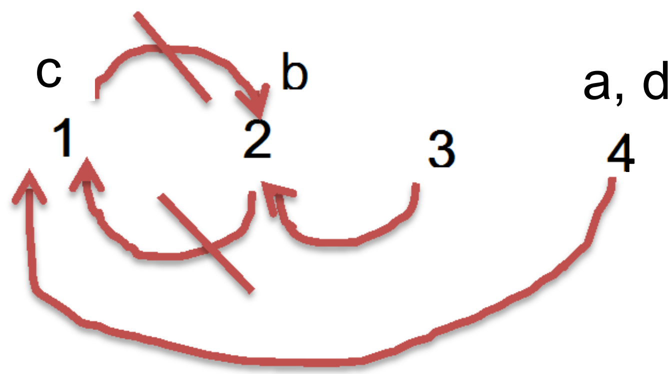
Example: **DTC** *rules are not Strategyproof*

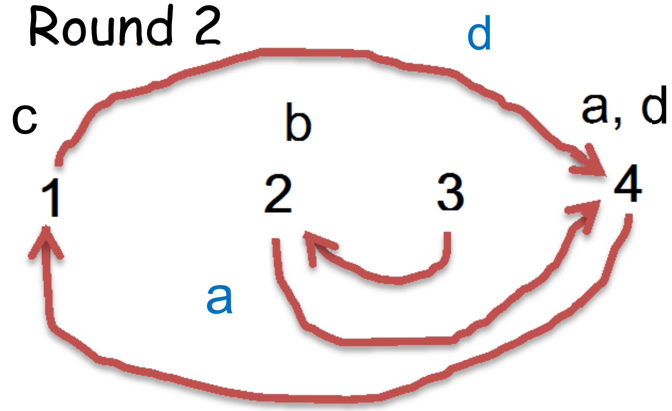
Agent 3 can manipulate:

3 can get object b instead of object d and bP_3d.

τ_a	τ_b	τ_c	τ_d
4	2	1	4
3	3	3	1
2	4	4	3
1	1	2	2

R_1	R_2	R'_3	R_4
b	c	b	c
d	a	d	b
		a	a
		c	





In the example, agent 3 is an intermediate agent for object b and 3 is "stalled" due to a tie for object c between agent 3 and agent 4. Agent 3 can manipulate here by pointing directly for object b, skipping c (i.e., by placing b above c in the preference ordering).

This is an example of an *intermediate agent cycle.*

INTERMEDIATE AGENT CYCLE

A weak priority table τ has an intermediate agent cycle if

- $\exists\, l, j, i \in N$ and $a, b \in M$ such that $l \pi_a i \, \tau_a j,\ \ j \pi_b l,\ \ j \pi_b i$
- $\exists\, h, h' \in N$ and $c \in M$ such that $h \, \tilde{\tau}_c h'$
- and one of the following two cases holds:

  case i): $h = i$ and $c \neq a$

  case ii): $h, h' \neq i$ and there is a priority chain from i to h

where l and j have a higher priority for a and b, respectively, than all the other concerned agents.

Strategyproofness of DTC rules

Call τ *intermediate acyclic (i-acyclic)* if it does not contain an intermediate agent cycle.

Strategyproofness of DTC rules

Call τ *intermediate acyclic (i-acyclic)* if it does not contain an intermediate agent cycle.

A DTC rule based on an i-acyclic weak priority table is an *Acyclic Deferred Trading Cycle rule.*

Strategyproofness of DTC rules

Call τ ***intermediate acyclic (i-acyclic)*** **if it does not contain an intermediate agent cycle.**

A DTC rule based on an i-acyclic weak priority table is an ***Acyclic Deferred Trading Cycle rule.***

All Acyclic Deferred Trading Cycle rules are Strategyproof.

Examples of Strategyproof **DTC** *rules*

- **τ is i-acyclic if it is all *strict*: DA rule**
- **τ is i-acyclic if it is all *weak:* TTC rule**

Examples of Strategyproof DTC rules

- τ is i-acyclic if it is all *strict*: DA rule
- τ is i-acyclic if it is all *weak:* TTC rule

A simple example of an i-acyclic weak priority table:

τ

a	b	c
1	2	3
2	1	1
3	3	2

R

1	2	3
c	a	a
b	b	c

R'

1	2	3
b	c	b
a	b	c

Matchings: DTC DA TTC

Theorem 2:

Individual Trade Stability

Constrained Efficiency

Lower Invariance

Strategyproofness

$\iff$ **Acyclic Deferred Trading Cycle Rules**

CONCLUSION

- How to combine some of the nice features of the DA and TTC rules without restricting the priority table?

  Is there a class of combined rules that captures the tradeoff between stability and efficiency and allows greater flexibility than using either the DA or the TTC rule?

  Deferred Trading Cycle Rules

  The weak priority table associated with the rule indicates where the simple priorities are enforced and hence shows the tradeoffs.

- *Are there combined rules which are strategyproof?*

Yes, the Acyclic Deferred Trading Cycle Rules