Roommate Market Core Characterizations

Bettina Klaus

Université de Lausanne

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Roommate Markets

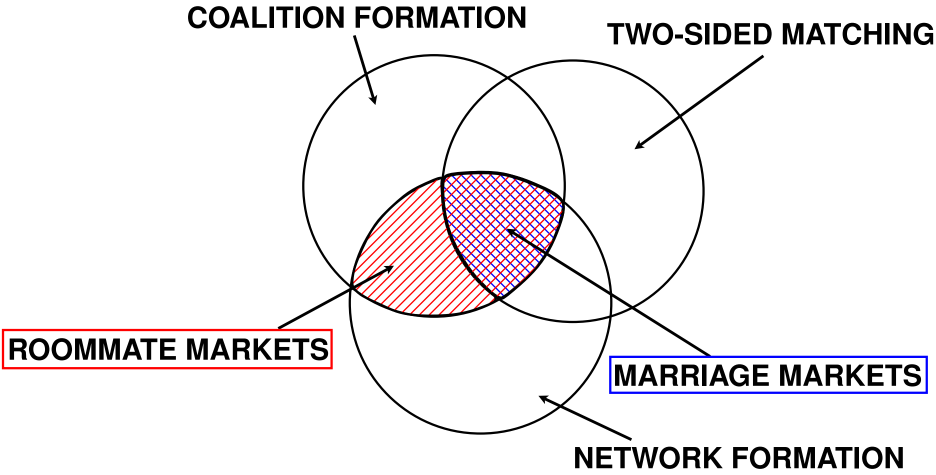
In their seminal paper Gale and Shapley (AMM 1962) introduced the very simple and appealing roommate problem as follows:

"An even number of boys wish to divide up into pairs of roommates."

A very common extension of this problem is to allow also for odd numbers of agents and to consider the formation of pairs and singletons (rooms can be occupied either by one or by two agents).

In addition, we will extend the problem to variable sets of agents, e.g., because the allocation of dormitory rooms at a university occurs every year for different sets of students.

Why Roommate Markets?



The Model I

- $\mathbb{N}$: set of potential agents.
- finite set of agents $N \subseteq \mathbb{N}$.
- R_i: agent i's strict preferences over sharing a room with any of the agents in $N \backslash \{i\}$ and having a room for himself.
- e.g.: $R_i : j, k, i, l, \ldots$
- (N, R): a *roommate market* .

The Model II

- A *marriage market* is a roommate market (N, R) such that N is the union of two disjoint sets M and W, and each agent in M (respectively W) prefers being single to being matched with any other agent in M (respectively W).

- A *matching* μ for roommate market (N, R) partitions the set of agents N into pairs and singletons.

- For classical marriage markets (Gale and Shapley, 1962), a matching never matches two men or two woman.

  We model marriage markets via preferences and not via the classical feasibility constraint on matchings.

- A *solution* φ is a correspondence that associates with each roommate market (N, R) a nonempty subset of matchings.

The Core

- *Stability*: the solution is *individually rational* and there exist *no blocking pairs* $\{i,j\}$ such that $j\,P_i\,\mu(i)$ and $i\,P_j\,\mu(j)$.

  Similarly as in other matching models (e.g., marriage markets and college admissions markets), the *core* equals the set of stable matchings.

- A roommate market is *solvable* if the set of stable matchings is non-empty.

  Gale and Shapley (1962) showed that all marriage markets are solvable (using *deferred acceptance*) and gave an example of an unsolvable roommate market.

A Roommate Market with an Empty Core

Example

Agent 1: 2 P_1 3 P_1 1,

Agent 2: 3 P_2 1 P_2 2,

Agent 3: 1 P_3 2 P_3 3.

In the example an odd ring (of length 3) is active.

An *odd ring* is an ordered set of agents $\{i_1, i_2, \ldots, i_k\} \subseteq N$, $k \geq 3$ odd, such that for all $t \in \{1, 2, .., k\}$,

$$i_{t+1} \; P_{i_t} \; i_{t-1} \; P_{i_t} \; i_t \text{ (modulo } k\text{)}.$$

A roommate market without an odd ring is called is called a *no odd rings roommate market* and is solvable.

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Agent 3: 1 P_3 **2** P_3 3.

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Normative Mechanism Design: “Nice Solutions?”

Two important papers are

- Sasaki and Toda (1992): “Consistency and Characterization of the Core of Two-Sided Matching Problems” and
- Toda (2006): “Monotonicity and Consistency in Matching Markets.”

To see in how far these results for marriage markets extend to roommate markets is the aim of:

- Özkal Sanver (2010): “Impossibilities for Roommate Problems,’
- Klaus (2011): “Competition and Resource Sensitivity in Marriage and Roommates Markets,”
- Can and Klaus (2012): “Consistency and Population Sensitivity Properties for Roommate Markets.”
- Klaus (2013):“Consistency and its Converse for Roommate Markets.”

Anonymity, (Converse) Consistency, and Pareto Optimality: Existing Results

Theorem (Sasaki and Toda, 1992)

On the domain of classical marriage markets with equal numbers of men and women and where all men find all women acceptable and all women find all men acceptable, a solution satisfies

- *anonymity*,
- *Pareto optimality*,
- *consistency*, and
- *converse consistency*

if and only if it is the *core*.

Anonymity, (Converse) Consistency, and Pareto Optimality: Existing Results

In a recent paper, Nizamogulari and Özkal-Sanver (2014) generalized the result to the full *domain of classical marriage markets* by

- adding *individual rationality* and
- replacing *anonymity* with "*gender fairness*."

However, on the domain of all roommate markets, *anonymity*, *Pareto optimality*, and *converse consistency* are not compatible (Özkal Sanver, 2010).[1]

Until recently it has been an open question if/how Toda and Sasaki's core characterization extend to the domain of solvable roommate markets and to the domain of no odd rings roommate markets.

[1] But the counterexample used in the proof is not solvable.

Anonymity, (Converse) Consistency, and Pareto Optimality: New Results

Theorem (Klaus, 2013)

On the domain of no odd rings roommate markets, a solution satisfies

- *individual rationality*,
- *anonymity*,
- *Pareto optimality*,
- *consistency*, and
- *converse consistency*

if and only if it is the *core*.

Remark

Our characterization of the core does not extend to the domain of solvable roommate markets. Example 1 in the paper describes a supersolution of the core satisfying all properties.

Unanimity, Consistency, and Monotonicity: Existing Results

Theorem (Toda, 2006)

For marriage markets, the *core* is the only solution satisfying

- *weak unanimity*,
- "*population monotonicity*," and
- *Maskin monotonicity*.

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For marriage markets, the *core* is the only solution satisfying

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- *consistency*.

Population Monotonicity: Comparing Sets

- Note that since we consider correspondences, agents who compare outcomes are comparing sets.

  To be more specific, let (N, R) and (N', R') be two roommate markets.

- We assume that agents are pessimistic and always assume that the worst matching will be realized, i.e., they compare the worst matching in $\varphi(N, R)$ to the worst matching in $\varphi(N', R')$.

  E.g., if for agent i the worst matching in $\varphi(N, R)$ is strictly better than the worst matching in $\varphi(N', R')$, then he prefers $\varphi(N, R)$ to $\varphi(N', R')$.

Population Change

A new set of agents, a set of newcomers $\tilde{N} \subset \mathbb{N} \backslash N$, shows up.

$$\Downarrow$$

$N' = N \cup \tilde{N}$ and extension (N', R') of (N, R).

Adding a set of agents $\tilde{N}$ might be a positive or a negative change for any of the incumbents in N because it might mean

Negative Change: more competition or

Positive Change: more resources.

Before we explore competition and resource sensitivity, we have a look at *population monotonicity* for marriage markets.

Population Monotonicity

- Population monotonicity is a solidarity property: additional agents enter $\Rightarrow$ all incumbents are affected in the same direction.

  This might not be a natural condition for marriage markets because of a certain polarization imbedded in the market: a man or a set of men might be considered good news for women (more choice), but bad news for men (more competition).

Definition (Own-Side Population Monotonicity for Marriage Markets)

A solution is *own-side population monotonic* if for any marriage market the following holds. If additional men [women] enter the market, then – because of the possible negative effect of the extra competition – all incumbent men [women] are weakly worse off.

Population Monotonicity

Definition (Other-Side Population Monotonicity for Marriage Markets)

A solution is *other-side population monotonic* if for any marriage market the following holds. If additional men [women] enter the market, then – because of the possible positive effect of the extra matching opportunities or resources – all incumbent women [men] are weakly better off.

Toda's population monotonicity is what we call own-side population monotonicity.

Competition Sensitivity

Without specifying whether newcomers are male or female, own-side population monotonicity implies that if $m, w \in N$ are new mates after the newcomers entered, then at least one of them is worse off (if the newcomers are all male, then man m is worse off and if the newcomers are all female, then woman w is worse off).

Definition (Competition Sensitivity)

A solution is *competition sensitive* if for any roommate market the following holds. If two incumbents are newly matched after a set of newcomers entered, then one of them suffers from the increased competition by the newcomers and is worse off.

Resource Sensitivity

Without specifying whether newcomers are male or female, other-side population monotonicity implies that if $m, w \in N$ are unmatched mates after the newcomers entered, then at least one of them is better off (if the newcomers are all male, then woman w is better off and if the newcomers are all female, then man m is better off).

Definition (Resource Sensitivity)

A solution is *resource sensitive* if for any roommate market the following holds. If two incumbents are unmatched after a set of newcomers entered, then one of them benefits from the increase of resources by the newcomers and is better off.

Characterizations of the Core: New Results

Theorem (Klaus, 2011)

On the class of marriage/no-odd-ring/solvable roommate markets, a solution φ satisfies

(a) *weak unanimity*, *competition sensitivity*, and *Maskin monotonicity*

(b) *weak unanimity*, *resource sensitivity*, and *Maskin monotonicity*

if and only if it equals the *core*.

Theorem (Can and Klaus, 2012)

On the class of marriage/no-odd-ring/solvable roommate markets, a solution φ satisfies

(a) *weak unanimity*, *competition sensitivity*, and *consistency*

(b) *weak unanimity*, *resource sensitivity*, and *consistency*

if and only if it equals the *core*.

Some Remarks

- The above theorems demonstrate that it is not really the full solidarity property (population monotonicity) that is at work in Toda's (2006) characterizations of the core for marriage markets, but that it is the population sensitivity property that is captured as well that is essential.
- We obtain impossibility results for the general domain of all roommate markets.